LINE-GEOMETRIC REPRESENTATIONS FOR FUNCTIONS OF A COMPLEX VARIABLE.*

BY

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INTRODUCTION

The usefulness of the usual method of analytic geometry, for visualizing a functional relation between two *real* variables, is too well known to call for further comment. A second method, not quite so familiar, has also been used extensively, especially in projective geometry. It consists in constructing scales on two straight lines, one for the independent and one for the dependent variable, joining corresponding points of the two scales by straight lines, and studying the one-parameter family of straight lines obtained in this way.

This second method can be utilized also for the purpose of visualizing a functional relation between two *complex* variables. If $z = x + iy$, and $w = u + iv$ are two complex variables, and if w is a function of z, we may proceed as follows. Choose any two planes of ordinary space as the planes of the two complex variables, and join by straight lines the points which corspond to each other in the functional relation. The result will be a two-parameter family of lines, a congruence, and this congruence may be regarded as a geometric image of the functional relation. It should be mentioned immediately, however, that the properties of this congruence depend, not merely on the properties of the function $w = F(z)$, but also upon the relative position of the two planes. We shall, in this paper, consider only the case which suggests itself most naturally, namely the case of two parallel planes. Only in this case will all of the points which represent finite values of z be geometrically distinct from those which represent finite values of w.

Weierstrass, in a very brief and incidental manner, mentions the possibility of such a line-geometric image of a functional relation.† But he did not investigate the properties of the resulting congruences; in fact he disposes of the whole matter in a very few words. There exists, however, an extensive

* Presented to the Society, December 28, 1918, and April 4, 1919.

† K. Weierstrass, *Mathematische Werke. Vorlesungen über die Theorie der Abel'schen Transcendenten*, Chapter 16, p. 323.

paper on this subject by M. J. Van Uven.* The work of Van Uven is devoted entirely to the discussion of special cases, such as $w = z^n$, $w = z^{-n}$, etc.; these are investigated in most thorough fashion, but the general questions are not attacked. A very interesting paper by Emch† makes use of a method which is applicable only to the case of a linear function. But, in spite of this limitation, Emch's paper is important, because in his paper the planes of the two complex variables are not parallel. Thus we are in a position to see, by comparing Emch's results with our own, that the mutual position of the two planes is an important feature of the theory. In fact we shall see that, for parallel planes, related to each other as described in Article 1, the congruence always has imaginary cones as focal surfaces, while Emch obtains real non-developable focal surfaces for the congruence which corresponds to the case which he has studied.

In Article 2 we shall present a second method of representing $w = F(z)$ by means of a congruence. This method consists in interpreting both variables as points of the same Riemann sphere, and then joining corresponding points by straight lines. The focal surfaces of the congruences obtained by this method are always real and possess some very striking properties. The possibility of such a representation seems to have occurred to Study,‡ but he did not develop the idea in detail. The method of Coolidge,§ which makes use of congruences in non-euclidean (elliptic) space, is different from any of these and far less elementary.

1. The method of parallel planes

Let us write

$$x + iy = z, \qquad x - iy = z_0, \qquad u + iv = w, \qquad u - iv = w_0, \tag{1}$$

and let us assume that w is a function of z, say

$$w = F(z), \tag{2}$$

so that

$$w_0 = F_0(z_0), \tag{3}$$

where F_0 denotes the function conjugate to F.

We interpret $z = x + iy$ in the usual way as a point P_z of the $\xi\eta$ plane, so

* M. J. Van Uven, *Algebraische Strahlencongruenzen und verwandte komplexe Ebenen als Schmitte derselben.* Verhandelingen der Konniklijke Akademie van Wetenschappen te Amsterdam (Eerste Sectie), vol. 10 (1910), pp. 1–527.

† A. Emch, *On the rectilinear congruence realizing a circular transformation of one plane into another*, Annals of Mathematics, ser. 2, vol. 13 (1911–12), p. 155.

Study, *Über Nicht-Euklidische und Linien-Geometrie*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 11 (1902), p. 329.

§ J. L. Coolidge, *Les congruences isotropes qui servent à représenter les fonctions d'une variable complexe*, Atti di Torino, vol. 38.

that the cartesian coördinates of P_z will be

$$\xi_1 = x, \qquad \eta_1 = y, \qquad \zeta_1 = 0.$$

We place the plane of the complex variable w parallel to that of z, in such a way that the origin of the w-plane shall be on the ζ-axis at the point $(0, 0, 1)$, the u- and v-axes being parallel to the x- and y-axes respectively. Then the cartesian coördinates of P_w will be

$$\xi_2 = u, \qquad \eta_2 = v, \qquad \zeta_2 = 1.$$

Let the *homogeneous* cartesian coördinates of P_z and P_w be $(\lambda_1, \cdots \lambda_4)$. and $(\mu_1, \cdots \mu_4)$ respectively. If we express x, y, u, and v in terms of z, z_0, w, w_0, by means of (1), we may write

$$\begin{aligned} &\lambda_1 = \frac{1}{2}(z + z_0), \qquad \lambda_2 = \frac{1}{2i}(z - z_0), \qquad \lambda_3 = 0, \qquad \lambda_4 = 1, \\ &\mu_1 = \frac{1}{2}(w + w_0), \qquad \mu_2 = \frac{1}{2i}(w - w_0), \qquad \mu_3 = 1, \qquad \mu_4 = 1, \end{aligned} \tag{4}$$

where, of course, only the ratios $\lambda_i : \lambda_k$ and $\mu_i : \mu_k$ are of significance.

In order to study the congruence, which is obtained when we join corresponding points P_z and P_w, we make use of a system of linear homogeneous differential equations satisfied by each of the four pairs of functions (λ_k, μ_k). This system of partial differential equations will be composed of two equations of the first order, and of two equations of the second order. The second order equations are obtained very easily. Obviously the equations

$$\frac{\partial^2 \lambda}{\partial z^2} = 0, \qquad \frac{\partial^2 \lambda}{\partial z_0^2} = 0, \qquad \frac{\partial^2 \mu}{\partial w^2} = 0, \qquad \frac{\partial^2 \mu}{\partial w_0^2} = 0 \tag{5}$$

are satisfied by each of the eight functions λ_k, μ_k. The latter two assume the form

$$\frac{\partial^2 \mu}{\partial z^2} - \frac{w''}{w'}\frac{\partial \mu}{\partial z} = 0, \qquad \frac{\partial^2 \mu}{\partial z_0} - \frac{w_0''}{w_0'}\frac{\partial \mu}{\partial z_0} = 0, \tag{6}$$

if we introduce z and z_0 as independent variables, and if we write

$$w' = \frac{dw}{dz}, \qquad w'' = \frac{d^2 w}{dz^2}, \qquad w_0 = \frac{dw_0}{dz_0}, \qquad w'' = \frac{d^2 w_0}{dz_0^2}.$$

In this connection the following remark is essential. If we maintain our original agreement that z and z_0 are conjugate complex variables, that is, that x and y are real, we can not treat z and z_0 as independent complex variables. Consequently we generalize our problem, at this stage, by thinking of x and y as independent complex variables. In most of the applications, however, we return to our original hypothesis, that x and y are real.

Direct substitution of the four pairs of functions (λ_k, μ_k) enables us to

verify that each of these pairs also satisfies the two first order equations

$$w'\frac{\partial\lambda}{\partial z}-\frac{\partial\mu}{\partial z}=0, \qquad w_0'\frac{\partial\lambda}{\partial z_0}-\frac{\partial\mu}{\partial z_0}=0. \tag{7}$$

Let us put

$$\bar{\lambda}=w'\lambda-\mu, \qquad \bar{\mu}=w_0'\lambda-\mu, \tag{8}$$

whence

$$(w'-w_0')\lambda=\bar{\lambda}-\bar{\mu}, \qquad (w'-w_0')\mu=w_0'\bar{\lambda}-w'\bar{\mu}. \tag{9}$$

Then we may write, in place of (7),

$$\frac{\partial\bar{\lambda}}{\partial z}=\frac{w''}{w'-w_0'}(\bar{\lambda}-\bar{\mu}), \qquad \frac{\partial\bar{\mu}}{\partial z_0}=\frac{w_0''}{w'-w_0'}(\bar{\lambda}-\bar{\mu}), \tag{10}$$

if we assume $w'-w_0'\neq 0$. The case thus excluded has little interest, since it corresponds to the case when w' is a real constant, so that $w=kz+b$ (k real). In this case the congruence degenerates into a bundle of lines.

We proceed to interpret equations (10). Under the assumption

$$w'-w_0'\neq 0,$$

equations (8) give rise to two distinct points, $P_{\bar{\lambda}}$ and $P_{\bar{\mu}}$, on the line $P_z P_w$. Let P_z move along one of the null lines $z_0=$ const. of the xy-plane. Then the line $P_z P_w$ generates an imaginary ruled surface of the congruence. The point $P_{\bar{\lambda}}$ describes a curve on this ruled surface whose tangent is obtained by joining $P_{\bar{\lambda}}$ to the point whose coördinates are $\partial\bar{\lambda}_k/\partial z$. According to (10) this tangent coincides with the line $P_{\bar{\lambda}}P_{\bar{\mu}}$, that is, with the line $P_z P_w$ itself. In other words, this ruled surface is a developable, and the locus of $P_{\bar{\lambda}}$ is its cuspidal edge. Moreover, since this developable contains a straight line (a null-line of the $\xi\eta$-plane), the cuspidal edge of the developable must be a plane curve, a fact which we shall also verify later analytically.

Thus, *the imaginary ruled surfaces of the congruence, which correspond to the two families of null lines of the $\xi\eta$-plane, constitute the developables of the congruence. Each of these developables touches one of the sheets of the focal surface along a plane curve.*

We can simplify the form of equations (10) by putting

$$f=\frac{\bar{\lambda}}{w'-w_0'}, \qquad g=\frac{\bar{\mu}}{w'-w_0'}. \tag{11}$$

They then assume the form

$$\frac{\partial f}{\partial z}=mg, \qquad \frac{\partial g}{\partial z_0}=nf, \tag{12}$$

where

$$m=\frac{-w''}{w'-w_0'}, \qquad n=\frac{+w_0''}{w'-w_0'}. \tag{13}$$

Since we are using homogeneous coördinates, f and g will represent the same

points as $\bar{\lambda}$ and $\bar{\mu}$ respectively, namely the two *foci* of a line of the congruence, that is, the two points in which $P_z P_w$ touches the two sheets of the focal surface.

By means of (9) and (11), we find

$$\lambda = f - g, \qquad \mu = w_0' f - w' g. \tag{14}$$

Equations (12) are the two first order differential equations, in the canonical form which is most convenient for the application of the general theory of congruences. The two second order equations may be obtained from (5) and (6) by introducing the variables f and g by means of (14). Two of the equations obtained in this way are

$$\frac{\partial^2 f}{\partial z_0^2} = \frac{\partial n}{\partial z_0} f + n \frac{\partial f}{\partial z_0}, \qquad \frac{\partial^2 g}{\partial z^2} = \frac{\partial m}{\partial z} g + m \frac{\partial g}{\partial z}; \tag{15}$$

the other two are consequences of (12) and (15).

Thus we have found that *our four pairs of functions* (f_k, g_k) *are solutions of a system of differential equations of the form**

$$\frac{\partial f}{\partial z} = mg, \qquad \frac{\partial g}{\partial z_0} = nf,$$

$$\frac{\partial^2 f}{\partial z_0^2} = af + bg + c \frac{\partial f}{\partial z_0} + d \frac{\partial g}{\partial z}, \tag{16}$$

$$\frac{\partial^2 g}{\partial z^2} = a' f + b' g + c' \frac{\partial f}{\partial z_0} + d' \frac{\partial g}{\partial z},$$

where

$$m = \frac{-w''}{w' - w_0'}, \qquad n = \frac{+w_0''}{w' - w_0},$$

$$a = \frac{\partial n}{\partial z_0}, \qquad b = 0, \qquad c = n, \qquad d = 0, \tag{17}$$

$$a' = 0, \qquad b' = \frac{\partial m}{\partial z}, \qquad c' = 0, \qquad d' = m.$$

In general, the locus of P_f, one of the sheets of the focal locus is a surface. It degenerates into a curve, if and only if m is equal to zero. This is an obvious consequence of (16), since f will then be a function of z_0 alone. But m can vanish identically, only if w is an integral linear function of z, and in that case n also will vanish, and therefore all of the coefficients of (16) will be equal to zero. To investigate the nature of the focal locus in this case, we make use of equations (4) and (8), after making the substitutions

$$w = \alpha z + \beta, \qquad w_0 = \alpha_0 z_0 + \beta_0,$$

* This is the form of the system upon which I have based the general projective theory of congruences. See *Sur la théorie générale des congruences*, Mémoires publies par la Classe des Sciences de l'Academie Royale de Belgique, 2 série, vol. 3 (1911). Hereafter quoted as *Brussels Paper*.

where α and β, α_0 and β_0, are constants, and where we shall assume $\alpha \neq \alpha_0$ so as to avoid the exceptional case $w' = w_0'$ whose significance has been pointed out already. A simple calculation shows that the focal locus consists of two straight lines, one through each of the circular points of the $\xi\eta$-plane.

In all other cases, m and n are different from zero, and the focal locus is composed of two surfaces. Equations (17) show that c' and d are equal to zero. Consequently *the focal surface of the congruence is composed of two developables.** Evidently the case $m = n = 0$ may be included in this statement, if we agree to regard a straight line as a degenerate case of a developable. Since we have, according to (15),

$$\frac{\partial^2 f}{\partial z_0} = \frac{\partial n}{\partial z_0} f + n \frac{\partial f}{\partial z_0}, \tag{18}$$

the curves on the sheet S_f of the focal surface, which correspond to the null lines $z = const.$ of the $\xi\eta$-plane, are straight lines. We have seen already that the curves on S_f which correspond to the null-lines $z_0 = const.$ are plane curves.

The first part of this theorem becomes quite obvious, if we remember that the two families of developables of the congruence must determine a conjugate system of curves on each sheet of the focal surface. In our case the focal surfaces are developable, and on a developable one of the component families of a conjugate system always consists of its generators.

If we put

$$h = \frac{\partial f}{\partial z_0} - nf, \tag{19}$$

and make use of (16) and (17), we find

$$\frac{\partial h}{\partial z} = 0, \qquad \frac{\partial h}{\partial z_0} = 0. \tag{20}$$

But clearly the point P_h, whose coördinates $(h_1, \cdots, h_4)$ are given by (19), is on that generator of the developable S_f which corresponds to a given constant value of z. According to (20), this point P_h is the same for all such generators; consequently *the focal surface S_f is a cone.* We may write

$$h = \frac{\partial f}{\partial z_0} - \frac{\partial g}{\partial z_0} = \frac{\partial \lambda}{\partial z_0}.$$

If we substitute the values (4) for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we find

$$h_1 : h_2 : h_3 : h_4 = i : -1 : 0 : 0. \tag{21}$$

Therefore, *the vertex of the focal cone S_f is one of the circular points of the $\xi\eta$-plane.*

The curves $z = \text{const.}$ of S_f are the generators of the cone. The curves $z_0 = \text{const.}$ of S_f are plane curves, a fact which we have mentioned before,

* Brussels Paper, p. 28.

but which may also be proved as follows. We have

$$\frac{\partial f}{\partial z} = mg, \qquad \frac{\partial^2 f}{\partial z^2} = \frac{\partial m}{\partial z} g + m \frac{\partial g}{\partial z},$$

$$\frac{\partial^3 f}{\partial z^3} = \left(\frac{\partial^2 m}{\partial z^2} + m \frac{\partial m}{\partial z} \right) g + \left(2 \frac{\partial m}{\partial z} + m^2 \right) \frac{\partial g}{\partial z}.$$

From these three equations, we may eliminate g and $\partial g/\partial z$. Thus we obtain the linear homogeneous differential equation of the third order for f;

$$(22) \quad \frac{\partial^3 f}{\partial z^3} - \left(m + 2 \frac{\partial \log m}{\partial z} \right) \frac{\partial^2 f}{\partial z^2} - \left[\frac{\partial^2 \log m}{\partial z^2} - \left(\frac{\partial \log m}{\partial z} \right)^2 \right] \frac{\partial f}{\partial z} = 0.$$

Since f_1, f_2, f_3, f_4 are solutions of this third order equation, the curves $z_0 = \text{const.}$ on S_f are indeed plane curves. Moreover equation (22) may serve for the purpose of discussing the projective properties of these curves, and therefore of the focal cone. But, unless we specialize the function $w = F(z)$, we need not expect to find any special properties of the focal cone beyond those already noted. In fact we shall show that this focal cone may be regarded as the most general analytic cone which has its vertex at $(i, -1, 0, 0)$.

To prove this we observe in the first place that the cone S_f is the envelope of the one-parameter family of planes

$$(23) \qquad x_1 + ix_2 + [z - F(z)]\, x_3 - zx_4 = 0,$$

all of which, of course, pass through the circular point $I\,(1, i, 0, 0)$, which is the vertex of the cone. Now the most general one-parameter family of planes through I may be represented by an equation of the form

$$(24) \qquad x_1 + ix_2 + \varphi(t) x_3 + \psi(t) x_4 = 0.$$

If $\psi(t)$ does not reduce to a constant, we may put $\psi(t) = -z$, and then determine $F(z)$ by means of the equation

$$z - F(z) = \varphi(t),$$

giving

$$F(z) = -\varphi(t) - \psi(t).$$

Thus, unless $\psi(t) = \text{const.}$, any family of form (24) is included in (23). If $\psi(t)$ is a constant, $\varphi(t)$ can not also be a constant if (24) is to represent a one-parameter family of planes. In the case $\psi(t) = \text{const.}$, (24) therefore represents a pencil of planes whose axis is a straight line in the xy-plane passing through I. This case only presents itself when z regarded as a function of w is a constant.

When the focal cone S_f is given, the second focal cone S_g is uniquely deter-

mined. It is the envelope of the one-parameter family of planes

$$x_1 - ix_2 + [z_0 - F_0(z_0)]\,x_3 - z_0\,x_4 = 0 \tag{25}$$

which are conjugate to (23).

Of course, the equations (23) and (25), regarded as simultaneous, are the equations of the two-parameter family of lines which constitute our congruence. In this form they are especially adapted to answer certain questions which arise naturally, especially when $F(z)$ is algebraic, as to the number of lines of the congruence which pass through a given point, or lie in a given plane, etc.

We may summarize our principal results as follows.

1. *If $w = F(z) = \text{const.}$, the congruence reduces to a bundle of lines whose vertex is the point $w = \text{const.}$ of the w-plane. Similarly if $z = F^{-1}(w) = \text{const.}$*

2. *If $w = kz + \beta$, where k and β are constants and where k is real, the congruence still reduces to a bundle of lines, but its vertex does not lie in the plane of either complex variable.*

3. *If $w = \alpha z + \beta$, where α and β are constants and where α is not real, the focal locus of the congruence is composed of two straight lines, one through each of the two circular points which the planes of z and w have in common.*

4. *In all other cases, the focal locus is composed of two cones whose vertices are the two circular points which the planes of z and w have in common. The developables of the congruence correspond to the null-lines of the two planes. Each of these developables is the locus of the tangents of a plane curve on one of the focal cones, and touches the other focal cone along a generator. The planes of the plane curves, thus determined on each of the focal cones, are the tangent planes of the other focal cone.*

Interesting as these results are, they do not help us to visualize the properties of the function $w = F(z)$, except in the simple cases 1 and 2, in which no such aid is needed. This is due to the fact that, in the general case, the developables and focal surfaces of the congruence are imaginary. In this respect the method of representation by means of the Riemann sphere (see Art. 2) is far superior. Nevertheless it is easy to utilize the method of parallel planes for furnishing real images of the functional relation. Thus every real curve of the z-plane determines a real ruled surface of the congruence which intersects the w-plane in a corresponding curve. These ruled surfaces, especially those which correspond to the lines $x = \text{const.}$ and $y = \text{const.}$, help us to understand the conformal correspondence between the two planes.* Moreover, certain loci connected with these ruled surfaces give rise to independent methods of defining the congruence.

We shall not, at present, discuss any questions of this general nature. We shall, however, indicate a method which enables us to replace the imaginary

* Van Uven's paper is largely devoted to the study of such ruled surfaces in the various special cases which he discusses.

focal surfaces by certain real surfaces very closely connected with them. The foci, P_f and P_g, of a line of the congruence are conjugate complex points of the line. Consequently the harmonic conjugate of P_z with respect to P_f and P_g will be a real point. We shall denote this real point by P_k, and the locus of P_k will be a surface S_k, whose properties we propose to investigate. Similarly we shall study the surface S_v, locus of P_v, where P_v is the harmonic conjugate of P_w with respect to P_f and P_g.

It follows from (14) that the homogeneous coördinates of P_k are given by

$$k = f + g. \tag{26}$$

We find

$$\frac{\partial k}{\partial z} = mg + \frac{\partial g}{\partial z}, \qquad \frac{\partial k}{\partial z_0} = nf + \frac{\partial f}{\partial z_0}, \tag{27}$$

and

$$\frac{\partial^2 k}{\partial z^2} = 2m_z g + 2mg_z, \qquad \frac{\partial^2 k}{\partial z_0^2} = 2n_{z_0} f + 2nf_{z_0},$$
$$\frac{\partial^2 k}{\partial z \partial z_0} = (m_{z_0} + mn)g + (n_z + mn)f, \tag{28}$$

where we have written

$$m_z = \frac{\partial m}{\partial z}, \qquad n_z = \frac{\partial n}{\partial z}, \qquad \text{etc.}$$

But, according to (17), we have

$$m_{z_0} = mn, \qquad n_z = mn, \tag{29}$$

so that we find

$$\frac{\partial^2 k}{\partial z \partial z_0} = 2mnk. \tag{30}$$

Consequently, *the curves on S_k, which correspond to the null-lines of the plane, form a conjugate system with equal Laplace-Darboux invariants.**

Moreover, *the curves of this conjugate system S_k are plane curves.*† For we have

$$\frac{\partial^3 k}{\partial z^3} = 2(m_{zz} + mm_z)g + 2(2m_z + m^2)g_z. \tag{31}$$

From this equation, and the equations for $\partial k/\partial z$ and $\partial^2 k/\partial z^2$, we may eliminate g and g_z, obtaining a linear homogeneous differential equation of the third order for the curve $z_0 = \text{const.}$ on S_k, viz.;

$$(m^2 - m_z)\frac{\partial^3 k}{\partial z^3} + (m_{zz} - mm_z - m^3)\frac{\partial^2 k}{\partial z^2} - 2(mm_{zz} - 2m_z^2)\frac{\partial k}{\partial z} = 0. \tag{32}$$

Only if $m_z = m^2$ will this plane curve reduce to a straight line. But we

* G. Darboux, *Théorie des surfaces*, vol. 2, p. 27.

† This remark was first made by one of my students, Dr. Cyril Nelson.

find, from (13),

$$m_z = m^2 + \frac{w'''}{w''}\, m, \qquad n_{z_0} = n^2 + \frac{w_0'''}{w_0''}\, n. \tag{33}$$

Therefore, if $m_z = m^2$, we must have either $m = 0$ or $w''' = 0$. The former case corresponds to $w = \alpha z + \beta$, where α and β are constants, and in this case S_k degenerates into a plane. If $m_z = m^2$, $m \neq 0$, we have $w''' = 0$, or $w = \alpha z^2 + \beta z + \gamma$, where α, β, and γ are constants, and $n_{z_0} = n^2$. Consequently we find, in this case,

$$k_{zz} = 2mk_z, \qquad k_{z_0 z_0} = 2nk_{z_0}, \qquad k_{zz_0} = 2mnk,$$

so that, in this case also, S_k degenerates into a plane,* and the curves $z = \text{const.}$ and $z_0 = \text{const.}$ in this plane are straight lines.

If we discard these cases, each of the curves $z = \text{const.}$ and $z_0 = \text{const.}$ of S_k, which intersect at P_k, determine a plane. The plane of the curve $z_0 = \text{const.}$ is determined by

$$k = f + g, \qquad \frac{\partial k}{\partial z} = mg + g_z, \qquad \frac{\partial^2 k}{\partial z^2} = 2m_z g + 2mg_z;$$

that of the curve $z = \text{const.}$ by

$$k = f + g, \qquad \frac{\partial k}{\partial z_0} = nf + f_{z_0}, \qquad \frac{\partial^2 k}{\partial z_0^2} = 2n_{z_0} f + 2nf_{z_0}.$$

Of course $k = f + g$ is a point of both planes. Since we may write

$$k_{zz} - 2mk_z = 2(m_z - m^2)g \tag{34}$$

and

$$k_{z_0 z_0} - 2nk_{z_0} - 2(n_{z_0} - n^2)k = -2(n_{z_0} - n^2)g, \tag{35}$$

we see that the point P_g is common to both planes, and therefore the same thing is true of P_f.

In other words; *the lines of our congruence are defined completely if the real surface S_k and the conjugate system of plane curves on S_k is known. They are the lines of intersection of the planes of these curves. Our congruence is the so-called axis congruence of this conjugate system on S_k.*† It is clear, moreover, that the planes of the curves on S_k which form the conjugate system are the tangent planes of the two focal cones.

In order to obtain a complete projective theory of the surface S_k, we must find the two second order partial differential equations which k_1, k_2, k_3, k_4 will satisfy. One of these is (30). The other one may be obtained from (34)

* E. J. Wilczynski, *One-parameter family of nets of plane curves.* These Transactions, vol. 12 (1911), p. 474.

† E. J. Wilczynski, *The general theory of congruences,* These Transactions vol. 16, (1915), pp. 312–317.

and (35) by eliminating g. It then appears in the form

$$(36)\qquad \begin{aligned}(n_{z_0}-n^2)(k_{zz}-2mk_z)+(m_z-m^2)(k_{z_0z_0}-2nk_{z_0})\\ =2(m_z-m^2)(n_{z_0}-n^2)k.\end{aligned}$$

But, on account of (13) and (33), this becomes

$$(37)\quad w_0'''k_{zz}-w'''k_{z_0z_0}+\frac{2w''w_0'''}{w'-w_0'}k_z+\frac{2w'''w_0''}{w'-w_0'}k_{z_0}+\frac{2w'''w_0'''}{w'-w_0'}k=0,$$

where we shall assume $w'-w_0'\neq 0$, $w'''\neq 0$.

This equation shows that the coefficients of the second fundamental form of S_k, with respect to the parameters z and z_0, are proportional to

$$w''',\qquad 0,\qquad w_0''',$$

so that the differential equation of the asymptotic lines of S_k is

$$w'''dz^2+w_0'''dz_0^2=0.$$

Consequently *the conjugate system formed on S_k by the curves $z=$ const. and $z_0=$ const. is an isothermally conjugate system.* The asymptotic lines of S_k are obtained by equating to constants the variables

$$R=\frac{1}{2}\Big[\int\sqrt{w'''}\,dz+i\int\sqrt{w_0'''}\,dz_0\Big],\qquad S=\frac{1}{2}\Big[\int\sqrt{w'''}\,dz-i\int\sqrt{w_0'''}\,dz_0\Big].$$

These variables are not real, but we shall show that they differ from real variables only by constant imaginary factors. For, if we put

$$\int\sqrt{w'''}\,dz=\lambda+i\mu,\qquad \int\sqrt{w_0'''}\,dz_0=\lambda-i\mu,$$

where λ and μ are real, we find

$$R=\frac{1+i}{2}(\lambda+\mu),\qquad S=\frac{1-i}{2}(\lambda-\mu).$$

Consequently the variables

$$r=\frac{1-i}{\sqrt{2}}R,\qquad s=-\frac{1+i}{\sqrt{2}}S$$

are real and constant along the asymptotic lines of S_k. We find

$$\begin{aligned}p&=r+is\\ &=\frac{1}{2\sqrt{2}}\Big[(1-i)\int\sqrt{w'''}\,dz+(1+i)\int\sqrt{w_0'''}\,dz_0\\ &\qquad -i(1+i)\int\sqrt{w'''}\,dz-(1+i)\int\sqrt{w_0'''}\,dz_0\Big],\end{aligned}$$

or

$$p = \frac{1-i}{\sqrt{2}} \int \sqrt{w'''}\, dz, \qquad p_0 = \frac{1+i}{\sqrt{2}} \int \sqrt{w_0'''}\, dz_0, \tag{38}$$

or, if we prefer,

$$p = \int \sqrt{iw'''}\, dz, \qquad p_0 = \int \sqrt{-iw_0'''}\, dz_0.$$

Since r and s are the components of the complex variable $p = r + is$, which is a function of $z = x + iy$, we see that *the asymptotic lines of S_k correspond to an orthogonal isothermal system of curves in the z-plane, namely to the curves obtained by equating to constants the real and the imaginary parts of the function*

$$p = \frac{1-i}{\sqrt{2}} \int \sqrt{w'''}\, dz.$$

Thus the asymptotic curves on S_k are real, and S_k is a surface of negative curvature.

If we introduce p and p_0 as independent variables, the two differential equations for k assume the form

$$\begin{aligned} &\frac{\partial^2 k}{\partial p^2} + \frac{\partial^2 k}{\partial p_0^2} + \frac{1+i}{\sqrt{2}} \frac{1}{\sqrt{w'''}} \left[\frac{2w''}{w' - w_0'} + \frac{1}{2} \frac{w^{(4)}}{w'''} \right] \frac{\partial k}{\partial p} \\ &\qquad - \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{w_0'''}} \left[\frac{2w_0''}{w' - w_0'} - \frac{1}{2} \frac{w_0^{(4)}}{w_0'''} \right] \frac{\partial k}{\partial p_0} + \frac{2i}{w' - w_0'} k = 0, \\ &\frac{\partial^2 k}{\partial p \partial p_0} + \frac{2w''\, w_0''\, k}{(w' - w_0')^2 \sqrt{w'''\, w_0'''}} = 0. \end{aligned} \tag{39}$$

Since $p = r + is$, $p_0 = r - is$, we obtain from this the differential equations of S_k referred to its asymptotic lines, namely

$$\begin{aligned} &\frac{\partial^2 k}{\partial r^2} + \frac{1}{2} \frac{W_r}{W} \frac{\partial k}{\partial r} - \frac{1}{2} \frac{W_s}{W} \frac{\partial k}{\partial s} + \left[\frac{4w''\, w_0''}{W} + \frac{2i}{w' - w_0'} \right] k = 0, \\ &\frac{\partial^2 k}{\partial s^2} - \frac{1}{2} \frac{W_r}{W} \frac{\partial k}{\partial r} + \frac{1}{2} \frac{W_s}{W} \frac{\partial k}{\partial s} + \left[\frac{4w''\, w_0''}{W} - \frac{2i}{w' - w_0'} \right] k = 0, \end{aligned} \tag{40}$$

where

$$W = (w' - w_0')^2 \sqrt{w'''\, w_0'''}. \tag{41}$$

We have already observed that we obtain a second real surface S_v as the locus of P_v where P_v is the harmonic conjugate of P_w with respect to P_f and P_g. Since the homogeneous coördinates of P_w are given by $\mu = w_0' f - w' g$, those of P_v will be

$$v = w_0' f + w' g. \tag{42}$$

We find

$$\frac{\partial v}{\partial z} = (w_0' m + w'')\, g + w' g_z,$$

$$\frac{\partial^2 v}{\partial z \partial z_0} = (w_0'' m + w_0' mn + w' mn)\, g + (w'' n + w_0' mn + w' mn) f,$$

which reduces to

$$\frac{\partial^2 v}{\partial z \partial z_0} = 2mnv. \tag{43}$$

Of course the properties of S_v, in relation to our congruence, are essentially the same as those of S_k, but equation (43) shows further that S_k and S_v *satisfy the same Moutard equation.*

2. The method of the Riemann sphere

We now pass to the second method mentioned in the introduction. We have, as before,

$$w = u + iv = F(z) = F(x + iy).$$

We use the $\xi\eta$-plane as the plane of both of the complex variables. Let P_z and P_w be two points of the $\xi\eta$-plane which represent a value of z and the corresponding value of w, and let P_1 and P_2 be the two points obtained from P_z and P_w, upon the sphere of unit radius and center at the origin, by stereographic projection from the point $(0, 0, 1)$. The coördinates of P_1 and P_2 will be

$$\xi_1 = \frac{z + z_0}{zz_0 + 1}, \qquad \eta_1 = \frac{-i(z - z_0)}{zz_0 + 1}, \qquad \zeta_1 = \frac{zz_0 - 1}{zz_0 + 1},$$

and

$$\xi_2 = \frac{w + w_0}{ww_0 + 1}, \qquad \eta_2 = \frac{-i(w - w_0)}{ww_0 + 1}, \qquad \zeta_2 = \frac{ww_0 - 1}{ww_0 + 1},$$

respectively. We may therefore write for the homogeneous cartesian coordinates of P_1,

$$\lambda_1 = z + z_0, \quad \lambda_2 = -i(z - z_0), \quad \lambda_3 = zz_0 - 1, \quad \lambda_4 = zz_0 + 1, \tag{44}$$

and for those of P_2,

$$\mu_1 = w + w_0, \quad \mu_2 = -i(w - w_0), \quad \mu_3 = ww_0 - 1, \quad \mu_4 = ww_0 + 1, \tag{45}$$

where, of course,

$$w = F(z), \qquad w_0 = F_0(z_0).$$

We propose to study the congruence generated by the line $P_1 P_2$ when $z = x + iy$ ranges over the domain in which the function $F(z)$ is defined. In order to apply the general theory of congruences we must find two linear partial differential equations of the first order, and two of the second order which are satisfied by the four pairs of functions (λ_k, μ_k). Since λ_k is linear in z, and μ_k is linear in w, the second order equations may be taken in the form

$$\frac{\partial^2 \lambda}{\partial z^2} = 0, \qquad \frac{\partial^2 \mu}{\partial z^2} - \frac{w''}{w'}\frac{\partial \mu}{\partial z} = 0, \tag{46}$$

the latter being derived from

$$\frac{\partial^2 \mu}{\partial w^2} = 0$$

by introducing z as independent variable. Of course we have also

$$\frac{\partial^2 \lambda}{\partial z_0^2} = 0, \qquad \frac{\partial^2 \mu}{\partial z_0^2} - \frac{w_0''}{w_0'} \frac{\partial \mu}{\partial z_0} = 0, \tag{47}$$

but these are consequences of (46) and the two first order equations which are still to be found. To obtain these we use the method of undetermined coefficients. We write

$$\begin{aligned} \frac{\partial \lambda}{\partial z_0} &= a_{11} \frac{\partial \lambda}{\partial z} + a_{12} \frac{\partial \mu}{\partial z} + b_{11} \lambda + b_{12} \mu, \\ \frac{\partial \mu}{\partial z_0} &= a_{21} \frac{\partial \lambda}{\partial z} + a_{22} \frac{\partial \mu}{\partial z} + b_{21} \lambda + b_{22} \mu, \end{aligned} \tag{48}$$

and determine the coefficients a_{ik} and b_{ik} by demanding that each of the equations (48) shall be satisfied when we put

$$\lambda = \lambda_k, \qquad \mu = \mu_k, \qquad (k = 1, 2, 3, 4).$$

We find

$$\begin{aligned} a_{11} &= 0, \qquad a_{12} = -\frac{z - w}{z_0 - w_0} \frac{dz}{dw}, \qquad b_{11} = -b_{12} = \frac{1}{z_0 - w_0}, \\ a_{21} &= -\frac{z - w}{z_0 - w_0} \frac{dw_0}{dz_0}, \qquad a_{22} = 0, \qquad b_{21} = -b_{22} = \frac{\dfrac{dw_0}{dz_0}}{z_0 - w_0}. \end{aligned} \tag{49}$$

Thus, the desired system of partial differential equations is the following:

$$\begin{aligned} \frac{\partial \lambda}{\partial z_0} &= -\frac{z - w}{z_0 - w_0} \frac{dz}{dw} \frac{\partial \mu}{\partial z} + \frac{1}{z_0 - w_0} (\lambda - \mu), \\ \frac{\partial \mu}{\partial z_0} &= -\frac{z - w}{z_0 - w_0} \frac{dw_0}{dz_0} \frac{\partial \lambda}{\partial z} + \frac{\dfrac{dw_0}{dz_0}}{z_0 - w_0} (\lambda - \mu), \\ \frac{\partial^2 \lambda}{\partial z^2} &= 0, \qquad \frac{\partial^2 \mu}{\partial z^2} - \frac{w''}{w'} \frac{\partial \mu}{\partial z} = 0. \end{aligned} \tag{50}$$

Equations (46) show that *the ruled surfaces $z_0 = const.$ of the congruence have two straight-line directrices.* That this must be so is, of course, evident. For, on account of the conformal character of the correspondence between z and w, and since the stereographic projection is also conformal, z and w will both describe null-lines of the $\xi\eta$-plane if either of them does, and then

the points P_1 and P_2 will simultaneously describe null-lines (imaginary generators) of the sphere.

The foci of the line $P_1 P_2$, that is the two points where the line touches the focal surface of the congruence, may be obtained by means of general methods familiar to geometers. As applied to a system of form (48), these methods show that the homogeneous coördinates of the foci are the factors of the quadratic covariant

$$a_{21}\lambda^2 - (a_{11} - a_{22})\lambda\mu - a_{12}\mu^2.$$

If we use (49), we find that, in our case, the foci are given by

$$\bar{\lambda} = \mu + \sqrt{w' w_0'}\,\lambda, \qquad \bar{\mu} = \mu - \sqrt{w' w_0'}\,\lambda, \tag{51}$$

where

$$w' = \frac{dw}{dz} = F'(z), \qquad w_0' = \frac{dw_0}{dz_0} = F_0'(z_0).$$

Since $w' w_0'$ is positive, and since λ_k and μ_k are real, we see that *the focal surfaces of this congruence are always real.* Moreover, equations (51) show, that *the foci of every line of the congruence separate harmonically the two points, P_1 and P_2, in which this line intersects the Riemann sphere.* Finally we observe that *the two foci of a line are distinct except upon those lines of the congruence which correspond to a solution of the equation $F'(z) = 0$ or of $F'(z) = \infty$.*

Thus the two sheets of the focal surface can hang together only at those points for which $F'(z)$ becomes zero or infinite.

The developables of the congruence may be obtained, in the general case of a system of form (48), by equating to constants two independent solutions, $\varphi(z, z_0)$ and $\psi(z, z_0)$, of the partial differential equation

$$\left(\frac{\partial\theta}{\partial z_0}\right)^2 - (a_{11} + a_{22})\frac{\partial\theta}{\partial z}\frac{\partial\theta}{\partial z_0} + (a_{11}a_{22} - a_{12}a_{21})\left(\frac{\partial\theta}{\partial z}\right)^2 = 0.*$$

In our case, using the values of a_{ik} given by (49), it is easy to perform the integration. We find in this way the following result.

The developables of our congruence are the ruled surfaces given by $\alpha = const.$ and $\beta = const.$, where

$$\alpha + i\beta = t = i\int\frac{\sqrt{w'}\,dz}{z - w}, \qquad \alpha - i\beta = t_0 = -i\int\frac{\sqrt{w_0'}\,dz_0}{z_0 - w_0}. \tag{52}$$

Since α and β are the components of a complex variable t which is a function of z alone, *the developables of the congruence correspond to an orthogonal isothermal system of curves on the Riemann sphere.*

* Here, as well as in the determination of the focal surfaces, I am making use of some formulæ from a theory of one-parameter families of ruled surfaces which is, as yet, unpublished. The reader, who finds it difficult to prove these general formulæ, should follow the transformations mechanically. The final results enable him to verify, *a posteriori*, the correctness of our statements.

We shall have to transform equations (50) into an equivalent system for the functions $\bar{\lambda}$ and $\bar{\mu}$ of α and β, before we can apply the formulæ of the general theory of congruences to our particular case. This transformation may be accomplished as follows.

From (51), we find

$$\lambda = \frac{1}{2\sqrt{w'w_0'}}(\bar{\lambda} - \bar{\mu}), \qquad \mu = \tfrac{1}{2}(\bar{\lambda} + \bar{\mu}). \tag{53}$$

If we substitute these values in (50), we find

$$\begin{aligned}
\frac{\partial\bar{\lambda}}{\partial z_0} &= -\sqrt{\frac{w_0'}{w'}}\frac{z-w}{z_0-w_0}\frac{\partial\bar{\lambda}}{\partial z} + \frac{1}{2(z_0-w_0)}\left(1+\sqrt{\frac{w_0'}{w'}}\right)[\bar{\lambda}-\bar{\mu} \\
&\quad - \sqrt{w'w_0'}(\bar{\lambda}+\bar{\mu})] + \frac{1}{4}\left[\frac{w_0''}{w_0'} + \frac{z-w}{z_0-w_0}\sqrt{\frac{w_0'}{w'}}\frac{w''}{w'}\right](\bar{\lambda}-\bar{\mu}), \\
\frac{\partial\bar{\mu}}{\partial z_0} &= +\sqrt{\frac{w_0'}{w'}}\frac{z-w}{z_0-w_0}\frac{\partial\bar{\mu}}{\partial z} - \frac{1}{2(z_0-w_0)}\left(1-\sqrt{\frac{w_0'}{w'}}\right)[\bar{\lambda}-\bar{\mu} \\
&\quad - \sqrt{w'w_0'}(\bar{\lambda}+\bar{\mu})] - \frac{1}{4}\left[\frac{w_0''}{w_0'} - \frac{z-w}{z_0-w_0}\sqrt{\frac{w_0'}{w'}}\frac{w''}{w'}\right](\bar{\lambda}-\bar{\mu}),
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
\frac{\partial^2\bar{\lambda}}{\partial z^2} - \frac{w''}{w'}\frac{\partial\bar{\lambda}}{\partial z} - \frac{1}{4}\{w,z\}(\bar{\lambda}-\bar{\mu}) &= 0, \\
\frac{\partial^2\bar{\mu}}{\partial z^2} - \frac{w''}{w'}\frac{\partial\bar{\mu}}{\partial z} + \frac{1}{4}\{w,z\}(\bar{\lambda}-\bar{\mu}) &= 0,
\end{aligned} \tag{55}$$

where $\{w, z\}$ is the Schwarzian derivative of w with respect to z, and where a definite determination must be chosen in every specific case of a branch of the function $\sqrt{w'}$; the value of $\sqrt{w_0'}$ will then be uniquely determined if we agree that $\sqrt{w'w_0'} = \sqrt{w'}\sqrt{w_0'}$ shall mean the positive value of this square root.

We continue the transformation by introducing the variables, t and t_0, defined by (52), in place of z and z_0. The result of this transformation on equations (54) is

$$\begin{aligned}
-i\sqrt{w'w_0'}\left(\frac{\partial\bar{\lambda}}{\partial t_0} - \frac{\partial\bar{\lambda}}{\partial t}\right) &= \\
&= +\left[\frac{1}{2}(\sqrt{w'}+\sqrt{w_0'})(1-\sqrt{w'w_0'})\right. \\
&\quad \left. + \frac{1}{4}\left\{(z_0-w_0)\sqrt{w'}\frac{w_0''}{w_0'} + (z-w)\sqrt{w_0'}\frac{w''}{w'}\right\}\right]\bar{\lambda} \\
&\quad - \left[\frac{1}{2}(\sqrt{w'}+\sqrt{w_0'})(1+\sqrt{w'w_0'})\right.
\end{aligned}$$

$$+\frac{1}{4}\left\{(z_0-w_0)\sqrt{w'}\frac{w_0''}{w_0'}+(z-w)\sqrt{w_0'}\frac{w''}{w'}\right\}\Big]\bar{\mu},$$

(56)
$$-i\sqrt{w'w_0'}\left(\frac{\partial\bar{\mu}}{\partial t_0}+\frac{\partial\bar{\mu}}{\partial t}\right)$$
$$=-\left[\frac{1}{2}(\sqrt{w'}-\sqrt{w_0'})(1-\sqrt{w'w_0'})\right.$$
$$\left.+\frac{1}{4}\left\{(z_0-w_0)\sqrt{w'}\frac{w_0''}{w_0'}-(z-w)\sqrt{w_0'}\frac{w''}{w'}\right\}\right]\bar{\lambda}$$
$$+\left[\frac{1}{2}(\sqrt{w'}-\sqrt{w_0'})(1+\sqrt{w'w_0'})\right.$$
$$\left.+\frac{1}{4}\left\{(z_0-w_0)\sqrt{w'}\frac{w_0''}{w_0'}-(z-w)\sqrt{w_0'}\frac{w''}{w'}\right\}\right]\bar{\mu}.$$

It will not be necessary to apply this transformation to (55) just now; we shall do this presently however in a slightly modified form.

Since $t=\alpha+i\beta$, $t_0=\alpha-i\beta$, we obtain, from (56), a system of the form

(57)
$$\frac{\partial\bar{\lambda}}{\partial\beta}=m'\bar{\lambda}+m''\bar{\mu},\qquad \frac{\partial\bar{\mu}}{\partial\alpha}=n'\bar{\lambda}+n''\bar{\mu},$$

where

$$4m'=2\left(\frac{1}{\sqrt{w'}}+\frac{1}{\sqrt{w_0'}}\right)(1-\sqrt{w'w_0'})$$
$$+(z_0-w_0)\frac{w_0''}{(w_0')^{3/2}}+(z-w)\frac{w''}{(w')^{3/2}},$$
$$4m''=-2\left(\frac{1}{\sqrt{w'}}+\frac{1}{\sqrt{w_0'}}\right)(1+\sqrt{w'w_0'})$$
$$-(z_0-w_0)\frac{w_0''}{(w_0')^{3/2}}-(z-w)\frac{w''}{(w')^{3/2}},$$

(58)
$$4in'=2\left(\frac{1}{\sqrt{w_0}}-\frac{1}{\sqrt{w'}}\right)(1-\sqrt{w'w_0'})$$
$$+(z_0-w_0)\frac{w_0''}{(w_0')^{3/2}}-(z-w)\frac{w''}{(w')^{3/2}},$$
$$4in''=-2\left(\frac{1}{\sqrt{w_0'}}-\frac{1}{\sqrt{w'}}\right)(1+\sqrt{w'w_0'})$$
$$-(z_0-w_0)\frac{w_0''}{(w_0')^{3/2}}+(z-w)\frac{w''}{(w')^{3/2}}.$$

A transformation of the form

$$\bar{\lambda}=\rho f,\qquad \bar{\mu}=\sigma g$$

can always be made so as to transform a system of form (57) into one of the binomial form

$$\frac{\partial f}{\partial\beta}=mg,\qquad \frac{\partial g}{\partial\alpha}=nf,$$

by imposing the conditions

$$\frac{\partial \rho}{\partial \beta} = m' \rho, \qquad \frac{\partial \sigma}{\partial \alpha} = n'' \sigma$$

upon ρ and σ. We can satisfy these conditions in our case by putting

$$\rho = \sigma = \sqrt[4]{w' w_0'}\sqrt{(z - w)(z_0 - w_0)}, \tag{59}$$

so that we may even choose ρ and σ as positive quantities. Consequently the transformation

$$f = \frac{\bar{\lambda}}{\sqrt[4]{w' w_0'}\sqrt{(z - w)(z_0 - w_0)}} = \frac{\sqrt[4]{w' w_0'}\,\lambda + \dfrac{\mu}{\sqrt[4]{w' w_0'}}}{\sqrt{(z - w)(z_0 - w_0)}},$$
$$g = \frac{\bar{\mu}}{\sqrt[4]{w' w_0}\sqrt{(z - w)(z_0 - w_0)}} = \frac{-\sqrt[4]{w' w_0'}\,\lambda + \dfrac{\mu}{\sqrt[4]{w' w_0'}}}{\sqrt{(z - w)(z_0 - w_0)}}, \tag{60}$$

furnishes the first order equations of our system in their final form

$$f_\beta = mg, \qquad g_\alpha = nf, \tag{61}$$

where $f_\beta = \partial f/\partial \beta$, etc., and where m and n, which are real functions of α and β, are defined by

$$m + in = -\frac{1}{\sqrt{w'}}\left[1 + w' + \frac{1}{2}(z - w)\frac{w''}{w'}\right], \tag{62}$$

whence

$$m - in = -\frac{1}{\sqrt{w_0'}}\left[1 + w_0' + \frac{1}{2}(z_0 - w_0)\frac{w_0''}{w_0'}\right]. \tag{63}$$

By means of (60) we have four pairs of functions (f_k, g_k) corresponding to the four pairs (λ_k, μ_k). Since the coördinates, which we are using, are homogeneous the points P_f and P_g coincide with the foci $P_{\bar{\lambda}}$ and $P_{\bar{\mu}}$ respectively.

In order to complete the transformation of our second order equations, we first use (60) to express $\bar{\lambda}$ and $\bar{\mu}$ in terms of f and g. The result of the substitution of these values in (55) is

$$\frac{\partial^2 f}{\partial z^2} - \frac{t''}{t'}\frac{\partial f}{\partial z} - \frac{1}{4}\left[\left(\frac{w''}{w'} - \frac{t''}{t'}\right)^2 + \frac{2w''}{z - w}\right]f + \frac{1}{4}\{w, z\}\, g = 0,$$
$$\frac{\partial^2 g}{\partial z^2} - \frac{t''}{t'}\frac{\partial g}{\partial z} + \frac{1}{4}\{w, z\}\, f - \frac{1}{4}\left[\left(\frac{w''}{w'} - \frac{t''}{t'}\right)^2 + \frac{2w''}{z - w}\right]g = 0, \tag{64}$$

whence

$$\frac{\partial^2 f}{\partial t^2} - \frac{1}{4(t')^2}\left[\left(\frac{w''}{w'} - \frac{t''}{t'}\right)^2 + \frac{2w''}{z - w}\right)\right]f + \frac{\{w, z\}}{4(t')^2}\, g = 0,$$
$$\frac{\partial^2 g}{\partial t^2} + \frac{\{w, z\}}{4(t')^2} f - \frac{1}{4(t')^2}\left[\left(\frac{w''}{w'} - \frac{t''}{t'}\right)^2 + \frac{2w''}{z - w}\right]g = 0 \tag{65}$$

where

$$t' = \frac{dt}{dz} = \frac{i\sqrt{w'}}{z-w}, \qquad \frac{t''}{t'} = -\frac{1-w'}{z-w} + \frac{1}{2}\frac{w''}{w'},$$

$$\frac{w''}{w'} - \frac{t''}{t'} = \frac{1-w'}{z-w} + \frac{1}{2}\frac{w''}{w'}. \tag{66}$$

Finally we find, from (65),

$$f_{\alpha\alpha} = af + bg + cf_\alpha + dg_\beta,$$

$$g_{\beta\beta} = a'f + b'g + c'f_\alpha + d'g_\beta, \tag{67}$$

where

$$a = 4 + n^2 - m^2, \qquad b = \frac{1}{4}\left[\frac{(z-w)^2}{w'}\{w,z\} + \frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\}\right] = -m_\beta,$$

$$c = 0, \qquad d = m, \tag{68}$$

$$a' = -b, \qquad b' = -a, \qquad c' = n, \qquad d' = 0.$$

Equations (61) and (67) constitute the system of partial differential equations of our congruence, in its canonical form. The second form, given here for the value of b, is an immediate consequence of the following relations which are useful in a large number of questions;

$$m_z = -\frac{1}{4}\frac{z-w}{\sqrt{w'}}\{w,z\}, \qquad m_{z_0} = -\frac{1}{4}\frac{z_0-w_0}{\sqrt{w_0'}}\{w_0,z_0\},$$

$$n_z = \frac{i}{4}\frac{z-w}{\sqrt{w'}}\{w,z\}, \qquad n_{z_0} = -\frac{i}{4}\frac{z_0-w_0}{\sqrt{w_0'}}\{w_0,z_0\},$$

$$m_t = +\frac{i}{4}\frac{(z-w)^2}{w'}\{w,z\}, \qquad m_{t_0} = -\frac{i}{4}\frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\},$$

$$n_t = +\frac{1}{4}\frac{(z-w)^2}{w'}\{w,z\}, \qquad n_{t_0} = +\frac{1}{4}\frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\}, \tag{69}$$

$$m_\alpha = \frac{i}{4}\left[\frac{(z-w)^2}{w'}\{w,z\} - \frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\}\right],$$

$$m_\beta = -\frac{1}{4}\left[\frac{(z-w)^2}{w'}\{w,z\} + \frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\}\right],$$

$$n_\alpha = -m_\beta, \qquad n_\beta = m_\alpha,$$

where the last two equations are merely expressive of the fact, already noted, that $m + in$ is a function of $x + iy$ and therefore of $\alpha + i\beta$.

The focal sheets of the congruence can degenerate into curves only if m or n, or both m and n, are equal to zero. Equations (69) show that this can happen

only if w is a linear function of z, say

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \tag{70}$$

where we may assume $\alpha\delta - \beta\gamma = 1$, since the case $w = \text{const.}$ excluded by this hypothesis, is of no interest. We find from (62), if we choose

$$\sqrt{w'} = (\gamma z + \delta)^{-1}, \qquad m + in = -(\alpha + \delta), \qquad m - in = -(\alpha_0 + \delta_0),$$

so that

$$2m = -(\alpha + \alpha_0 + \delta + \delta_0), \qquad 2in = -(\alpha - \alpha_0 + \delta - \delta_0).$$

Consequently we obtain the following result.

The sheet S_f of the focal surface is degenerate if and only if w is a linear function of z of the form

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma = 1,$$

for which the real parts of α and δ are numerically equal but opposite in sign. The second sheet S_g degenerates if and only if the corresponding relation holds between the imaginary parts of α and δ, that is, if α and δ are conjugate complex. Both sheets degenerate if and only if*

$$\alpha + \delta = 0.$$

In all other cases, even if w is a linear function of z,† the focal surfaces are non-degenerate. Equations (68) show that

$$W = mn - c'd = 0$$

in all cases. Consequently *the congruence is a W-congruence,‡ that is, it has the property of making the asymptotic lines of the two sheets of the focal surface correspond to each other.* That this is so, also follows directly from equations (55). For we may regard (55) as the differential equations of a ruled surface $z_0 = \text{const.}$ of our congruence. The form of the equations shows immediately that the curves of contact of this ruled surface with the two sheets of the focal surface are asymptotic curves upon both of them.§ Of course the same facts may be read off from equations (64) or (65).

These simple remarks enable us to conclude further that the ruled surfaces

* If besides, β and $-\gamma$ are conjugate complex, (70) reduces to a rotation of the Riemann sphere.

† If w is a linear function, aside from these exceptional cases, both sheets of the focal surface are quadrics.

‡ Brussels paper, p. 34 and p. 46.

§ E. J. Wilczynski, *Projective differential geometry of curves and ruled surfaces*, Leipzig (1906), p. 142.

$z_0 = \text{const.}$ and $z = \text{const.}$, that is, *the ruled surfaces of the congruence which are determined by the imaginary generators of the Riemann sphere are the ones which touch the focal surface along its asymptotic lines.*

Since these ruled surfaces are imaginary, while the focal surface is real, *both sheets of the focal surface must have positive curvature everywhere.* But the consideration of these ruled surfaces teaches us still more. The curve of contact with either sheet of the focal surface is an asymptotic curve, not only of the focal surface, but of the ruled surface as well. But this ruled surface has two straight line directrices, namely the two imaginary generators of the Riemann sphere which the points P_1 and P_2 describe when z_0 remains constant. Every asymptotic line on a ruled surface with two straight line directrices has the property that each of its asymptotic lines belongs to a linear complex, that is, there exists a linear complex which contains all of its tangents. It follows then that *every asymptotic line on each sheet of the focal surface of our congruence belongs to a linear complex.*

We wish to verify this conclusion analytically, and also to study some other questions which arise in this connection. If we eliminate g and g_β from equations (61) and (67) we find

$$\begin{aligned} f_{\alpha\alpha} - f_{\beta\beta} &= -2\frac{m_\beta}{m} f_\beta + af, \\ f_{\alpha\beta} &= \frac{m_\alpha}{m} f_\beta + mnf, \end{aligned} \tag{71}$$

the differential equations of the focal sheet S_f referred to the curves $\alpha = \text{const.}$ and $\beta = \text{const.}$, which are the curves along which the developables of the congruence touch S_f. The second of these equations merely states the familiar fact that the curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ form a conjugate system on S_f; the first equation of (71) shows that *this system of curves on S_f is isothermally conjugate.*

Let us transfrom (71) by introducing $t = \alpha + i\beta$ and $t_0 = \alpha - i\beta$ as independent variables. We find a system of differential equations of the form

$$\begin{aligned} \frac{\partial^2 f}{\partial t^2} + 2A\frac{\partial f}{\partial t} + 2B\frac{\partial f}{\partial t_0} + Cf = 0, \\ \frac{\partial^2 f}{\partial t_0^2} + 2A'\frac{\partial f}{\partial t} + 2B'\frac{\partial f}{\partial t_0} + C'f = 0, \end{aligned} \tag{72}$$

where

$$\begin{aligned} A &= -\frac{1}{2}\frac{\partial \log m}{\partial t}, \quad B = +\frac{1}{2}\frac{\partial \log m}{\partial t}, \quad C = \frac{1}{4}[(m+in)^2 - 4], \\ A' &= +\frac{1}{2}\frac{\partial \log m}{\partial t_0}, \quad B' = -\frac{1}{2}\frac{\partial \log m}{\partial t_0}, \quad C' = \frac{1}{4}[(m-in)^2 - 4]. \end{aligned} \tag{73}$$

Equations (72) are the differential equations of the surface S_f referred to its asymptotic lines.* From (62), (63), and (73) we find

$$\frac{\partial^2 \log A'}{\partial t \partial t_0} = \frac{\partial^2 \log B}{\partial t \partial t_0} = 4A'B. \tag{74}$$

But these equations constitute the analytic verification, which we were seeking; they show, according to C. T. Sullivan, that the asymptotic curves of S_f belong to linear complexes.† Sullivan has also noted that every osculating ruled surface of such a surface has two straight-line directrices, and that the locus of these directrices is a quadric surface, which he calls the directrix quadric.‡ It is evident, from what precedes, that in our case *the directrix quadric associated in this way with each of the two sheets of the focal surface is the Riemann sphere.*

Let us consider an arbitrary point P of the surface S_f. The osculating linear complexes of the two asymptotic curves of S_f, which cross at P, have a linear congruence in common. The directrices of this congruence are called the directrices of the point P; one of them passes through P while the other lies in the plane tangent to S_f at P. The resulting two congruences have the property that their developables correspond to each other and to a common net of curves on S_f called its directrix curves.§ For a system of form (72), the differential equation of the directrix curves is

$$BLdt^2 + 2Mdtdt_0 - A'N\,dt_0^2 = 0,$$

where

$$L = -2A'(2A'BF + 2A'BB_{t_0} + BA'_{tt}) + BA'^2_t,$$

$$M = A'^2 B^2 \frac{\partial^2 \log (B/A')}{\partial t \partial t_0},$$

$$N = -2B(2A'BG + 2A'BA'_t + A'B_{tt_0}) + A'B^2_{t_0}.\|$$

In our case this differential equation reduces to

$$[4 - (m + in)^2]\,dt^2 - [4 - (m - in)^2]\,dt_0^2 = 0. \tag{75}$$

Consequently *the directrix curves on S_f form a conjugate system and its equations can be obtained by quadratures.* This property however is not peculiar to our case; it is common to all surfaces whose asymptotic lines belong to linear

* E. J. Wilczynski, *Projective differential geometry of curved surfaces.* (First Memoir), these Transactions, vol. 8 (1907), p. 244.

† C. T. Sullivan, *Properties of surfaces whose asymptotic lines belong to linear complexes,* these Transactions, vol. 13 (1912), p. 175.

‡ L. c., p. 185.

§ E. J. Wilczynski, *Projective differential geometry of curved surfaces* (Second Memoir), these Transactions, vol. 9 (1908), p. 95.

‖ L. c., p. 116, equations (136) and (137).

complexes.* But if we set up the differential equation for the directrix curves on S_g we obtain (75) over again. Consequently *the directrix curves of the two sheets of the focal surface correspond to each other.* This property is not known to be an obvious consequence of the fact that our congruence is a W-congruence. On account of this latter property, the system of curves on S_g which corresponds to the directrix curves of S_f must be a conjugate system, but it need not therefore be composed of the directrix curves of S_g.

The integrals of (75) are

$$\int \sqrt{4-(m+in)^2}\,dt \pm \int \sqrt{4-(m-in)^2}\,dt_0 = \text{const.};$$

but these equations are equivalent to those obtained by equating to constants the real and imaginary parts of the function

$$\int \sqrt{4-(m+in)^2}\,dt.$$

If we substitute for $m+in$ and dt their values, we obtain the following result.

The directrix curves on both sheets of the focal surface correspond to the same orthogonal isothermal system of curves of the $\xi\eta$-plane or on the sphere, namely to the system obtained by equating to constants the real and imaginary parts of the function

$$i\int \sqrt{4w' - \left[1 + w' + \frac{z-w}{2}\frac{w''}{w'}\right]^2}\,\frac{dz}{z-w}. \tag{76}$$

From this it follows, in particular, that *these directrix curves are real.* The same thing follows from the differential equation of the directrix curves when written in the form

$$mnd\alpha^2 - ad\alpha d\beta - mnd\beta^2 = 0, \tag{77}$$

obtained from (75) by introducing the real variables α and β in place of t and t_0. For, the coefficients and variables in (77) are real, and the discriminant

$$a^2 + 4m^2n^2 = (4 - m^2 + n^2)^2 + 4m^2n^2 \tag{78}$$

is positive.

The osculating planes of the curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ which meet at a point P_f of S_f intersect in a line through P_f which is called the *axis* of P_f. The axes of all points of S_f form a congruence called *axis congruence,* and the curves on S_f, which correspond to the developables of this congruence, are called its *axis curves.* The differential equation of the axis curves on S_f are

$$\left(mn - \frac{\partial^2 \log m}{\partial\alpha\partial\beta}\right)d\alpha^2 - \left(a - \frac{m_{\alpha\alpha}}{m}\right)d\alpha d\beta - mnd\beta^2 = 0.\dagger \tag{79}$$

* Sullivan, l.c., p. 194.

† E. J. Wilczynski, *The general theory of congruences,* these Transactions, vol. 16 (1915), p. 316, equation (9).

The axis curves on S_g are given by

$$(80) \qquad -mnd\alpha^2 + \left(a + \frac{n_{\beta\beta}}{n}\right) d\alpha d\beta + \left(mn - \frac{\partial^2 \log n}{\partial\alpha\partial\beta}\right) d\beta^2 = 0.$$

The Laplace transformation associates with every point of S_f two points in the plane tangent to S_f at P_f, namely the first and minus first Laplace transforms of P_f. The line joining these points is called the *ray* of P_f, and the totality of the rays of all points of S_f is called its *ray congruence*. The *ray curves* of S_f, that is, the curves of S_f which correspond to the developables of the ray congruence are given by

$$(81) \qquad -mnd\alpha^2 + \left(a - \frac{m_{\alpha\alpha}}{m}\right) d\alpha d\beta + \left(mn - \frac{\partial^2 \log m}{\partial\alpha\partial\beta}\right) d\beta^2 = 0.*$$

Similarly we find as differential equation of the ray curves on S_g, the following:

$$(82) \qquad \left(mn - \frac{\partial^2 \log n}{\partial\alpha\partial\beta}\right) d\alpha^2 - \left(a + \frac{n_{\beta\beta}}{n}\right) d\alpha d\beta - mnd\beta^2 = 0.$$

The asymptotic curves on S_f or S_g are given by

$$(83) \qquad dtdt_0 = d\alpha^2 + d\beta^2 = 0.$$

Consequently, a relation of the form

$$d\alpha_1\, d\alpha_2 + d\beta_1\, d\beta_2 = 0$$

between two tangents of S_f at P_f expresses the property of conjugacy. Clearly this relation exists between the surface tangents given by (79) and (81), and also between (80) and (82).

Therefore our congruence has the following additional property. *On each sheet of the focal surface, each of its axis curve tangents is conjugate to one of the ray-curve tangents.*

In order to study these relations a little more thoroughly we propose to find the equations of the directrices, axes, and rays of the points P_f and P_g, referred to a common coördinate system. The most convenient system of coördinates for this purpose is one whose fundamental tetrahedron is made up of the four points P_f, P_g, P_ρ, and P_σ, where P_ρ and P_σ are defined by

$$(84) \qquad \rho = f_\alpha - \frac{m_\alpha}{m} f, \qquad \sigma = g_\beta - \frac{n_\beta}{n} g;$$

P_ρ and P_σ are the first and minus first Laplace transforms of P_f and P_g with respect to the conjugate systems determined on S_f and S_g respectively by the developables of the congruence.

* L.c., p. 318, equation (23).

By direct application of the formulæ of the general theory of congruences we find the following results.*

The equations of the axis of P_f are

$$x_3 = 0, \qquad x_2 + \left(\frac{m_\beta}{m} - \frac{n_\beta}{n}\right) x_4 = 0; \tag{85}$$

the axis of P_g is given by

$$x_4 = 0, \qquad x_1 - \left(\frac{m_\alpha}{m} - \frac{n_\alpha}{n}\right) x_3 = 0. \tag{86}$$

The ray of P_f has the equations

$$x_1 = x_4 = 0, \tag{87}$$

and the ray P_g is given by

$$x_2 = x_3 = 0. \tag{88}$$

To find the equations of the directrices of P_f and P_g referred to this coördinate system is not quite so easy. In the projective theory of surfaces, the equations of the directrices of P_f are referred to a coördinate system whose vertices are given by†

$$\begin{gathered} f, \qquad \rho_f = f_t - \frac{1}{2}\frac{m_t}{m} f, \qquad \sigma_f = f_{t_0} - \frac{1}{2}\frac{m_{t_0}}{m} f, \\ \tau_f = f_{tt_0} - \frac{1}{2}\frac{m_{t_0}}{m} f_t - \frac{1}{2}\frac{m_t}{m} f_{t_0} + \left[\frac{1}{4}\frac{m_t m_{t_0}}{m^2} - \frac{1}{4}\left(\frac{m_t}{m}\right)_{t_0} - \frac{1}{4}\left(\frac{m_{t_0}}{m}\right)_t\right] f, \end{gathered} \tag{89}$$

and it becomes necessary to obtain the equations of transformation between the two systems, which may be called the local coördinate systems of the *surface* S_f and of the *congruence*. We find the following relations

$$\begin{aligned} \rho_f &= \frac{1}{4m}(m_\alpha + i m_\beta) f - \frac{i}{2} m g + \frac{1}{2}\rho, \\ \sigma_f &= \frac{1}{4m}(m_\alpha - i m_\beta) f + \frac{i}{2} m g + \frac{1}{2}\rho, \\ \tau_f &= \left[1 + \frac{1}{4}(n^2 - m^2) - \frac{1}{16}\frac{m_\alpha^2}{m^2} + \frac{3}{16}\frac{m_\beta^2}{m^2}\right] f + \frac{1}{4}\left(2m\frac{n_\beta}{n} - m_\beta\right) g \\ &\qquad - \frac{1}{4}\frac{m_\alpha}{m}\rho + \frac{1}{2} m\sigma. \end{aligned} \tag{90}$$

Let x_1, x_2, x_3, x_4 be the coördinates of a point with respect to the system f, g, ρ, σ, so that the expression

$$x_1 f + x_2 g + x_3 \rho + x_4 \sigma$$

* L. c., equations (5).

† First Memoir, p. 248, equations (40).

will furnish the homogeneous coördinates of the point referred to our fundamental homogeneous cartesian system when f, g, ρ, σ are replaced by f_t, g_t, ρ_t, σ_t, $(t = 1, 2, 3, 4)$. Similarly, let x_1', x_2', x_3', x_4' be the homogeneous coördinates of the same point referred to the local coördinate system of the surface S_f. Then the equation

$$\omega'(x_1 f + x_2 g + x_3 \rho + x_4 \sigma) = x_1' f + x_2' \rho_f + x_3' \sigma_f + x_4' \tau_f,$$

must be valid for all values of f, g, ρ, σ after ρ_f, σ_f, τ_f have been replaced by their values (90); in this equation ω' is an arbitrary factor of proportionality whose value is of no particular importance, on account of the homogeneity of our coördinate systems, and which may be given a special value so as to simplify the formulæ.

We obtain in this way the following equations of transformation:

$$\begin{aligned}
\omega' x_1 &= x_1' + \frac{1}{4}\frac{m_\alpha + i m_\beta}{m} x_2' + \frac{1}{4}\frac{m_\alpha - i m_\beta}{m} x_3' \\
&\qquad + \left(1 + \frac{1}{4}n^2 - \frac{1}{4}m^2 - \frac{1}{16}\frac{m_\alpha^2}{m^2} + \frac{3}{16}\frac{m_\beta^2}{m^2}\right) x_4', \\
\omega' x_2 &= -\frac{im}{2} x_2' + \frac{im}{2} x_3' + \frac{1}{4n}(2m n_\beta - n m_\beta) x_4', \\
\omega' x_3 &= \frac{1}{2} x_2' + \frac{1}{2} x_3' - \frac{1}{4}\frac{m_\alpha}{m} x_4', \\
\omega' x_4 &= \frac{1}{2} m x_4'.
\end{aligned} \tag{91}$$

In computing the inverse of this transformation it is convenient to put $\omega' = 2m$. We find

$$\begin{aligned}
x_1' &= 2m x_1 + \frac{m_\beta}{m} x_2 - m_\alpha x_3 + \left(m^2 - n^2 - 4 - \frac{1}{4}\frac{m_\alpha^2 + m_\beta^2}{m^2} - \frac{m_\beta n_\beta}{mn}\right) x_4, \\
x_2' &= 2i x_2 + 2m x_3 + \left(\frac{m_\alpha + i m_\beta}{m} - 2i\frac{n_\beta}{n}\right) x_4, \\
x_3' &= -2i x_2 + 2m x_3 + \left(\frac{m_\alpha - i m_\beta}{m} + 2i\frac{n_\beta}{n}\right) x_4, \\
x_4' &= 4 x_4.
\end{aligned} \tag{92}$$

Now, the equations of the directrix of the first kind of P_f are

$$x_4' = 0, \qquad x_1' + \frac{1}{2}\frac{A_t'}{A'} x_2' + \frac{1}{2}\frac{B_{t_0}}{B} x_3' = 0, \tag{93}$$

if we use the notations (73), and those for the directrix of the second kind are

$$x_2' + \frac{1}{2}\frac{B_{t_0}}{B}x_4' = 0, \qquad x_3' + \frac{1}{2}\frac{A_t'}{A}x_4' = 0.^* \tag{94}$$

Making use of (92) we find

$$x_4 = 0, \qquad x_1 - \frac{m_\alpha}{m}x_3 = 0 \tag{95}$$

as equations of the directrix of the first kind of S_f, and

$$x_3 = 0, \qquad x_2 - \frac{n_\beta}{n}x_4 = 0 \tag{96}$$

for the directrix of the second kind of S_f, both referred to the local coördinate system of the congruence.

In order to obtain the corresponding equations for the directrices of P_g, we must supply some formulæ for the surface S_g which are analogous to certain corresponding formulæ for S_f, but which we have not noted so far. The equations of S_g referred to the curves α = const. and β = const. are

$$\begin{aligned} g_{\alpha\alpha} - g_{\beta\beta} &= 2\frac{n_\alpha}{n}g_\alpha + ag, \\ g_{\alpha\beta} &= mng + \frac{n_\beta}{n}g. \end{aligned} \tag{97}$$

The equations of S_g referred to its asymptotic curves are

$$\begin{aligned} g_{tt} - \frac{n_t}{n}g_t - \frac{n_t}{n}g_{t_0} + \frac{1}{4}[(m+in)^2 - 4]g = 0, \\ g_{t_0t_0} - \frac{n_{t_0}}{n}g_t - \frac{n_{t_0}}{n}g_{t_0} + \frac{1}{4}[(m-in)^2 - 4]g = 0. \end{aligned} \tag{98}$$

The fundamental local coördinate system of S_g is given by the four expressions

$$\begin{aligned} g, \qquad \rho_g = g_t - \frac{1}{2}\frac{n_t}{n}g, \qquad \sigma_g = g_{t_0} - \frac{1}{2}\frac{n_{t_0}}{n}g, \\ \tau_g = g_{tt_0} - \frac{1}{2}\frac{n_{t_0}}{n}g_t - \frac{1}{2}\frac{n_t}{n}g_{t_0} + \frac{3}{4}\frac{n_t n_{t_0}}{n^2}g, \end{aligned} \tag{99}$$

so that

$$\rho_g = \frac{1}{2}nf - \frac{n_\alpha + in_\beta}{4n}g - \frac{i}{2}\sigma,$$

$$\sigma_g = \frac{1}{2}nf - \frac{n_\alpha - in_\beta}{4n}g + \frac{i}{2}\sigma,$$

* Second Memoir, p. 95, equations (70a) and (70b).

$$\tau_g = \frac{n}{4}\left(2\frac{m_\alpha}{m} - \frac{n_\alpha}{n}\right)f - \left(1 + \frac{1}{4}n^2 - \frac{1}{4}m^2 - \frac{3}{16}\frac{n_\alpha^2}{n^2} + \frac{1}{16}\frac{n_\beta^2}{n^2}\right)g + \frac{1}{2}n\rho - \frac{1}{4}\frac{n_\beta}{n}\sigma. \tag{100}$$

If the coördinates of a point, $x_1 f + x_2 g + x_3 \rho + x_4 \sigma$, referred to this system are x_1'', x_2'', x_3'', x_4'', we find

$$\begin{aligned}
\omega'' x_1 &= \frac{1}{2}nx_2'' + \frac{1}{2}nx_3'' + \frac{n}{4}\left(2\frac{m_\alpha}{m} - \frac{n_\alpha}{n}\right)x_4'',\\
\omega'' x_2 &= x_1'' - \frac{n_\alpha + in_\beta}{n}x_2'' - \frac{n_\alpha - in_\beta}{n}x_3''\\
&\qquad - \left(1 + \frac{1}{4}n^2 - \frac{1}{4}m^2 - \frac{3}{16}\frac{n_\alpha^2}{n^2} + \frac{1}{16}\frac{n_\beta^2}{n^2}\right)x_4'',\\
\omega'' x_3 &= \frac{1}{2}nx_4'',\\
\omega'' x_4 &= \frac{-i}{2}x_2'' + \frac{i}{2}x_3'' - \frac{1}{4}\frac{n_\beta}{n}x_4'',
\end{aligned} \tag{101}$$

whence

$$\begin{aligned}
x_1'' &= \frac{n_\alpha}{n}x_1 + 2nx_2 + \left(4 + n^2 - m^2 - \frac{n_\alpha^2 + n_\beta^2}{4n^2} - \frac{m_\alpha n_\alpha}{mn}\right)x_3 - n_\beta x_4,\\
x_2'' &= 2x_1 + \left(\frac{n_\alpha + in_\beta}{n} - 2\frac{m_\alpha}{m}\right)x_3 + 2nix_4,\\
x_3'' &= 2x_1 + \left(\frac{n_\alpha - in_\beta}{n} - 2\frac{m_\alpha}{m}\right)x_3 - 2nix_4,\\
x_4'' &= 4x_3.
\end{aligned} \tag{102}$$

From these formulæ we find

$$x_3 = 0, \qquad x_2 - \frac{n_\beta}{n}x_4 = 0 \tag{103}$$

as the equations of the directrix of the first kind of P_g, and

$$x_4 = 0, \qquad x_1 - \frac{m_\alpha}{m}x_3 = 0 \tag{104}$$

for the directrix of the second kind of P_g. Comparison with (95) and (96) furnishes the following theorem:

The two focal sheets of our congruence are so related that the directrix of the first kind belonging to any point of either sheet is at the same time the directrix of the second kind of the corresponding point of the other sheet.

THE UNIVERSITY OF CHICAGO,
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ON THE COMBINATION OF NON-LOXODROMIC SUBSTITUTIONS*

BY

EDWARD B. VAN VLECK

In a brief but interesting communication on *Finite groups of linear substitutions*† Vivanti notes that a necessary condition that the product of two elliptic substitutions

$$z' = \frac{az + b}{cz + d}$$

be elliptic is that their four poles shall lie upon a common circle. Finding this to be precisely the necessary and sufficient condition that their product be non-loxodromic, I was led to the consideration of the necessary and sufficient condition that the product of *any* two non-loxodromic substitutions of given character—i.e., hyperbolic, parabolic, or elliptic—should be a non-loxodromic substitution, or, more restrictedly, one of specified character. The results are of very simple nature and make a sort of geometric multiplication table for non-loxodromic substitutions with non-loxodromic products.

The sufficiency of the conditions given for non-loxodromic products is involved in the structure of the familiar non-loxodromic groups, and it is probable also that their necessity, though not stated, is well known to many workers in the field of automorphic functions. It does not appear, however, that a systematic study of the combination of non-loxodromic substitutions and of the specific nature of their product has been previously made, and various facts brought out seem to be altogether new. Special reference may be made to the criterion given in Theorem 4 for the character of the non-loxodromic product. The combination of the substitutions affords a basis for the formation of non-loxodromic groups by synthesis.‡

1. Preliminary considerations

In the following paper we shall consider exclusively non-loxodromic substitutions

$$z' = \frac{az + b}{cz + d} \qquad (S). \tag{1}$$

* Presented to the Society, April, 1916.

† Rendiconti del Circolo di Palermo, vol. 35 (1913), p. 160.

‡ A rapid analytic derivation of these groups is given in Fubini's *Introduzioni alla teoria dei gruppi discontinui e delle funzioni automorfe*, pp. 189–193. His treatment is closer to that of the following pages than anything else that I have found.

These we shall assume to be always taken in unimodular form with determinant $\Delta \equiv ad - bc$ *equal to* 1. Then the necessary and sufficient condition that S shall be non-loxodromic is that $a + d$ shall be real. It is hyperbolic, parabolic, or elliptic according as $|a + d|$ is greater than, equal to, or less than 2.

The character of S is not affected when it is transformed into $T^{-1}ST$ through any other linear substitution T. Any substitution with two distinct poles may be transformed into the form*

$$z' = k^2 z \qquad (S_1). \tag{2}$$

For a hyperbolic substitution k is real, while for an elliptic substitution it is an imaginary $e^{2i\phi}$ of unit modulus. Let now S_1 be followed by any other non-loxodromic substitution S. Their product $S'' = SS_1$ may be written in unimodular form

$$z' = \frac{akz + bk^{-1}}{ckz + dk^{-1}}, \qquad (S'') \tag{3}$$

and will be non-loxodromic if $ak + dk^{-1}$ is real, its character depending upon $|ad + dk^{-1}|$. Since the same is true of $S_1 S$, *the character of the resulting substitution is independent of the order of the combination of* S *and* S_1.

Unless $c = 0$, the poles of S are given by

$$z' = \frac{(a - d) \pm \sqrt{(d + a)^2 - 4}}{2c}, \tag{4}$$

those of S'' by

$$z' = \frac{(ak - dk^{-1}) \pm \sqrt{(dk^{-1} + ak)^2 - 4}}{2ck}, \tag{5}$$

while those of $S_1 S$ differ from (5) only by the substitution of $2ck^{-1}$ for $2ck$ in the denominator. When $c = 0$, the substitutions S, S_1, and S'' have a common pole at infinity.

If our unimodular substitution (1) is real, it is necessarily non-loxodromic. Its two poles then lie upon the real axis if $|a + d| \geqq 2$, that is, when the substitution is hyperbolic or parabolic, while they will be symmetrically situated with respect to this axis when it is elliptic.

Suppose, conversely, that the poles of a hyperbolic or elliptic unimodular substitution are situated respectively in this manner. Then the radical in the numerator of (4) is either real or a pure imaginary, and therefore $a - d$ must be real to permit the poles of S to have the assumed positions. Then a, d, and c are real and hence the substitution must be real. If the substitution is parabolic, we must suppose in the converse, not merely that its pole lies upon the real axis, but also that this axis is an invariant curve of

* In unimodular form, $z' = kz/k^{-1}$.

the substitution. Then the four coefficients of the unimodular substitution are real except possibly for a common factor. As $a + d = \pm 2$, this factor must be real. *Thus a unimodular substitution will be real if and only if it is a hyperbolic substitution with poles on the real axis, an elliptic substitution with poles symmetrically situated with respect to the axis, or a parabolic substitution having this axis as an invariant curve.*

For the sake of completeness we will include the trivial case of the combination of the two non-loxodromic substitutions having a pole in common. The conditions under which they furnish a non-loxodromic product are seen at once after throwing this pole to infinity. The two substitutions then take the form (not necessarily unimodular)

$$z' = k^2 z + a \quad (S_3), \qquad z' = k_1^2 z + b \quad (S_4).$$

Obviously they generate a non-loxodromic product and group unless one substitution is elliptic and the other hyperbolic. In this group there is always included a parabolic substitution

$$z' = z + a(1 - k_1^2) - b(1 - k^2) \qquad (S_3 S_4 S_3^{-1} S_4^{-1}).$$

This reduces to the identical substitution only if $a(1 - k_1^2) = b(1 - k^2)$, which is precisely the condition that S_3 and S_4 shall have their second pole in common.

These considerations make it clear that we have the following types of non-loxodromic groups with a single pole common to all the substitutions of the group, and only these types.

1. An elliptic-parabolic group of substitutions with one common pole.
2. A hyperbolic-parabolic group with one common pole.
3. A parabolic group whose substitutions have a common pole.

In addition, we have

4. Elliptic groups with both poles common.
5. Hyperbolic groups with both poles common.

There remains for consideration our main problem, that of the combination of two non-loxodromic substitutions without a common pole. If one is elliptic or hyperbolic, we may take it in the form (2) without loss of generality. Then $a + d$ and $ak + dk^{-1}$ are both real, provided the product of the two substitutions is non-loxodromic. Accordingly *if S_1 is hyperbolic, a and d are real simultaneously with k.* On the other hand, *if S_1 is elliptic* and accordingly k an imaginary of unit modulus, *a and d must be a pair of conjugate imaginaries.* In either case, ad and bc are real.

2. Combination with a hyperbolic substitution

Take first for consideration the combination of a hyperbolic substitution (2) with another non-loxodromic substitution (1) so as to give a non-loxodromic

product. As just stated, k, a, d, and bc must be real. By a turn of axes $z = e^{i\theta} w$ we do not alter the substitution (2) and merely replace b and c in (1) by $be^{-i\theta}$ and $ce^{i\theta}$. Hence we may make our substitution (1) take a real form. Then our two substitutions give rise to a group of real unimodular substitutions which are all non-loxodromic. Hence if S is hyperbolic (parabolic), its poles (pole) lie with the poles 0 and ∞ of S_1 upon the real axis which is an invariant line, while they are symmetrically situated with respect to this axis if S is elliptic. Conversely, by Section 1, if these conditions are fulfilled, S must be a real unimodular substitution, and hence generates with S_1 a non-loxodromic group of substitutions whose poles are disposed in similar fashion. When S and S_1 are transformed through any linear substitution T, our result takes the following form.

THEOREM 1. *The necessary and sufficient condition that the product of two hyperbolic substitutions be non-loxodromic is that their four poles shall be concyclic.* If a hyperbolic substitution is combined with a parabolic one, the corresponding condition is that the circle containing their three poles shall be invariant for the parabolic component. If a hyperbolic and elliptic substitution are combined, the condition for the formation of a non-loxodromic product is that there exists a circle upon which lie the poles of the hyperbolic component and with respect to which the poles of the elliptic component are symmetrically situated (i.e., are inverse points). In each case the group generated is non-loxodromic.*

In combining our hyperbolic substitution S_1 with another non-loxodromic substitution the existence of a non-loxodromic product was sufficient, as noted, to ensure that the group generated should be non-loxodromic. When, however, it is combined with more than one non-loxodromic substitution, the non-loxodromic character of their simple products is no longer sufficient to ensure the non-loxodromic character of the entire group. Consider the combination of S and S_1 with a third unimodular non-loxodromic substitution

$$z' = \frac{a'z + b'}{c'z + d'} \qquad (S') \tag{6}$$

so as to generate a non-loxodromic group. In Section 1 we disposed of the case in which all three substitutions have a pole in common. If S_1 has only one pole in common with S and the other in common with S', it can not have the former pole in common with SS' since this pole would then belong also to $S^{-1}(SS') = S'$, neither can it have its other pole in common with SS' since this pole would then belong also to $(SS')S'^{-1} = S$. Now SS' can be taken in place of either S or S' as one of the generators of the group. It will be therefore no restriction to suppose that one of our two substitutions to be combined with S_1 has neither pole common with S_1. Let this substitution be S.

* The condition holds true when the poles are not distinct. See Section 1.

As before, we will take S in real form. Its combination with (6) gives a product

$$z' = \frac{(aa' + b'c)z + (a'b + b'd)}{(c'a + d'c)z + (bc' + dd')} \qquad (S'S). \tag{7}$$

In both (6) and (7) the first and last coefficients must be real inasmuch as (6) and (7) by hypothesis combine with S_1 so as to produce non-loxodromic products. Consequently $b'c$ and bc' are real. Now neither b nor c is zero inasmuch as the poles of S_1 and S are distinct. It follows that b', c' are both real and hence the third substitution S' is real. The group generated by the three substitutions consists therefore of real unimodular substitutions which are all non-loxodromic.

Other non-loxodromic substitutions can clearly be added as generators of our non-loxodromic group provided that they also are real and unimodular. It has thus been shown that if a given set of non-loxodromic substitutions include a hyperbolic substitution (transformed above into form S_1), they will generate a non-loxodromic group if and only if they are capable of being transformed simultaneously into real unimodular form. In the italicized sentence of Section 1 the condition that a substitution should have such form was connected with the position of its poles relatively to the real axis. By transformation of our substitutions through any arbitrary substitution T the real invariant axis becomes a circle. We obtain then the following result.

THEOREM 2. *If a set of non-loxodromic substitutions without a common pole includes at least one hyperbolic substitution, the necessary and sufficient condition that they generate a non-loxodromic group is the existence of a common invariant circle, with respect to which the poles of the elliptic generators must be symmetrically situated. Upon this circle lie the poles of the hyperbolic and parabolic generators.*

It may be noted incidentally that this theorem admits all non-loxodromic substitutions which leave the circle invariant with the exception of elliptic substitutions of period 2 whose poles lie upon the circle.

Consider next the particular case of two hyperbolic substitutions (1) and (2) whose poles lie on a straight line through the origin and separate one another. A glance at (4) shows that the necessary and sufficient condition for this separation is

$$|(a + d)^2 - 4| > |a - d|^2. \tag{8}$$

Since $|a + d| > 2$, this reduces to the condition $ad > 1$ or its equivalent $bc > 0$. Then

$$|ak + dk^{-1}| > |ak + a^{-1}k^{-1}| > 2,$$

and the products $S'' = SS_1$ and S_1S are accordingly hyperbolic. Furthermore, the poles of S'' are separated by those of its first factor S_1 since

$ak \cdot dk^{-1} > 1$. For like reason the poles of $S''^{-1} = S_1^{-1} S^{-1}$ are separated by those of S^{-1}. Hence the poles of S'' are separated by those of each of its factors. Thus we get the following theorem.

THEOREM 3. *When the poles of two hyperbolic substitutions lie on a common circle and separate one another, the product of the two substitutions is hyperbolic, and its poles lie upon the same circle and are separated by those of each factor.*

It may be pointed out that the group generated by the two substitutions will contain pairs of hyperbolic substitutions whose poles do not separate one another. In particular, the poles of SS_1 are not separated by those of $S_1 S$ since they lie on opposite sides of the origin and differ by the positive factor k^{-2} from the poles of $S_1 S$.

Consider next the case of two hyperbolic substitutions whose poles lie upon a common circle but do not separate one another. Their product may be either hyperbolic, parabolic, or elliptic. To determine which, I shall derive a general criterion for the character of the product of two non-loxodromic substitutions whose poles are distinct and which have a non-loxodromic product. Expressed in terms of its two poles γ' and δ', our substitution (1) may be written

$$\frac{z' - \gamma'}{z' - \delta'} = k'^2 \left(\frac{z - \gamma'}{z - \delta'} \right), \tag{9}$$

whence we have

$$z' = \frac{(\gamma' - \delta' k'^2) z + \gamma' \delta' (k'^2 - 1)}{(1 - k'^2) z - (\delta' - \gamma' k'^2)}, \tag{10}$$

in which the four coefficients are to be divided by $k'(\gamma' - \delta')$ in order to make the substitution unimodular. A comparison of (10) with (1) gives the values

$$a = \frac{\gamma' - \delta' k'^2}{k'(\gamma' - \delta')}, \qquad d = -\frac{(\delta' - \gamma' k'^2)}{k'(\gamma' - \delta')}.$$

Since the poles γ, δ of S_1 have the special position 0, ∞, the ratio γ'/δ' which appears in these expressions is the anharmonic ratio

$$A = (\gamma' \delta' \gamma \delta)$$

of the four poles of S and S_1. If now these values of a and d are substituted in the inequalities

$$|ak + dk^{-1}| \gtreqless 2$$

determining the character of the product $S'' = SS_1$, the inequalities may be written in the form (11) below. Now neither the anharmonic ratio A nor the characteristic multipliers k^2, k'^2 of S_1 and S are changed in value when S and S_1 are transformed through any linear substitution T. We obtain therefore the following useful criterion for the character of the product transformation.

THEOREM 4. *If the four poles of two non-loxodromic substitutions are distinct and their product is non-loxodromic, the product will be hyperbolic, parabolic, or elliptic according as*

$$|1 + k^2 k'^2 - A(k^2 + k'^2)| \gtreqless 2|kk'| \cdot |1 - A|. \tag{11}$$

Here A denotes the anharmonic ratio $(\gamma' \delta' \gamma\delta)$ of the poles γ, δ and γ', δ' of the two substitutions, and k^2, k'^2 are the corresponding characteristic multipliers of the substitutions (as indicated in (9)).

The anharmonic ratio of four points in the z-plane is real only when the four points are concyclic. Consider now the case of two hyperbolic substitutions with concyclic poles. The constants k and k' are real, and A is negative when the two pairs of poles separate one another. The upper sign will then hold in the inequality (11), and the product of the two hyperbolic substitutions is accordingly hyperbolic, as already stated in Theorem 3. Suppose, on the other hand, that A is positive so that the two pairs of poles do not separate one another. The sign of equality in (11) will hold if

$$1 + k^2 k'^2 - A(k^2 + k'^2) = \pm 2kk'(1 - A),$$

that is, when

$$A = \frac{(1 + kk')^2}{(k + k')^2} \quad \text{or} \quad A = \frac{(1 - kk')^2}{(k - k')^2}.$$

The product of the two hyperbolic substitutions is then parabolic. It is easy to see that when A lies between these two positive values the product is elliptic, while if A is positive and does not lie between these limits, the product is hyperbolic. In the latter case the poles of the product $S'' = SS_1$ or of $S''' = S_1 S$ can not be separated by the poles of either of its factors. For if they were separated by those of S, the application of Theorem 3 to $S_1 = S^{-1} S''$ and $S_1 = S''' S^{-1}$ respectively would make the poles of S and S_1 separate one another, contrary to hypothesis. For like reason they can not be separated by the poles of S_1.

Consider lastly the combination of a hyperbolic substitution S_1 with an elliptic substitution S whose poles γ', δ' are taken to lie symmetrically with respect to the real axis (or any line through the origin) so as to furnish a non-loxodromic product. We then have $|A| = |\gamma'|/|\delta'| = 1$, and clearly this condition is also sufficient for the existence of such a product. Hence the necessary and sufficient condition that an elliptic and hyperbolic substitution should combine to give a non-loxodromic product is that the anharmonic ratio $(\gamma' \delta' \gamma\delta)$ of their four poles should be of unit modulus.

3. COMBINATION WITH A PARABOLIC SUBSTITUTION

Consider next the combination of a parabolic substitution P with any other non-loxodromic substitution S. Without loss of generality we may suppose

the parabolic substitution to be taken in the form $z' = z + h$ where h is real. In our substitution (1) $a + d$ must be real, and c can not be zero since the two substitutions would then have a common pole. The product PS has the form

$$z' = \frac{az + b}{cz + d} + h = \frac{(a + ch)z + b + dh}{cz + d}$$

and will be non-loxodromic only if $a + d + ch$ is real, that is, if c is real. By taking n sufficiently large we may make $|a + d + nch| > 2$, and the group generated by the two substitutions must therefore contain a hyperbolic substitution $H = P^n S$. Now $PS \equiv P^{-(n-1)} H$, and accordingly the product PS will be non-loxodromic under the conditions of Theorem 1 applied to the combination of a hyperbolic substitution H and the parabolic substitution $P^{-(n-1)}$. But since P and P^{n-1} have the same system of invariant circles, the conditions of the theorem are fulfilled by P and H (which generate the same group as P and S) at the same time as by P^{n-1} and H. Hence by Theorems 1 and 2, we have immediately the following result.

THEOREM 5. *If two non-loxodromic substitutions are without a common pole and one of them is parabolic, the necessary and sufficient condition that they shall have a non-loxodromic product is the existence of a common invariant circle, with the added requirement that the poles of the second substitution must be symmetrically situated with respect to the circle if it is elliptic. The two substitutions then generate a non-loxodromic group in which hyperbolic substitutions are necessarily included.*

4. Combination of elliptic substitutions

It remains now only to consider the combination of two non-loxodromic substitutions (1) and (2) which are both elliptic. When they have a non-loxodromic product, a and d are conjugate imaginaries with $|a + d| < 2$, as noted in Section 1, while $k = e^{i\phi}$. Accordingly the two terms in the numerator of (4) are pure imaginaries, and the poles of S lie with the poles of S_1 on a straight line through the origin. Conversely, suppose that the four poles of the two substitutions lie on the same straight line, necessarily through the origin. Since $a + d$ is real and less than 2 in absolute value, the two terms in the numerator of (4) must both be pure imaginaries; hence a and d are conjugate imaginaries. Then $ak + dk^{-1}$ is real and the product of the two substitutions is non-loxodromic. Thus the necessary and sufficient condition for a non-loxodromic product is that their four poles shall lie on a straight line. By transformation of the two substitutions through an arbitrary substitution T this becomes the requirement that *the four poles shall be concyclic.*

If $|ak + dk^{-1}| < 2$, the product $S'' = SS_1$ is elliptic. Inasmuch as the products $S'' S_1^{-1} = S$ and $S^{-1} S'' = S_1$ are elliptic, the poles of S'' must be

concyclic with the poles of S, also with the poles of S_1. Now the two terms in the numerator of (5) as well as (4) are pure imaginaries, and a comparison of their denominators shows that the poles of S'' lie on a line through the origin which makes an angle $-\arg k = -\phi$ with the line containing the poles of S_1 (or $+\phi$ if the order of the two factors is reversed). When the product is parabolic, the two poles of S'' come together on this line. Lastly, if $|ak + dk^{-1}| > 2$, the numerator of (5) has the form $iA \pm B$, so that the poles of the hyperbolic product S'' are symmetrically situated with respect to the same line.

These results give the following theorem.

THEOREM 6. *The necessary and sufficient condition that the product of two elliptic substitutions be non-loxodromic is that their four poles shall lie upon a common circle. When the product is elliptic, its two poles are concyclic with the poles of each factor, and the circle containing the poles of S_1 and $S'' = SS_1$ makes an angle $-\phi$ with the circle containing the poles of S_1 and S, if the multiplier of S_1 is $k^2 = e^{2i\phi}$ (or $+\phi$, if the order of the two factors is reversed). The two poles of S'' come together on the former circle when S'' is parabolic and are symmetrically situated with respect to the circle when it is hyperbolic.*

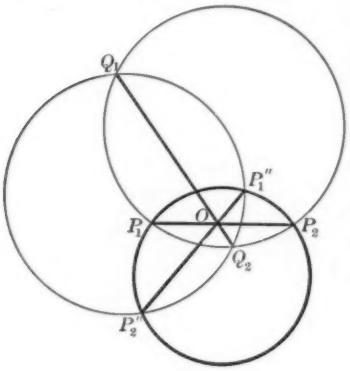
Two cases will now be distinguished according as the poles of S and S_1 do or do not separate one another. The condition for the separation is given in (8). Since a and d are now conjugate imaginaries with $|a + d| < 2$, this reduces to the inequality $ad < 1$ or its equivalent $bc < 0$. This makes $|a| \equiv |d| < 1$ and $|ak + dk^{-1}| < 2$, and therefore the product of the two substitutions is elliptic when their poles separate one another. Further, the poles P_1'', P_2'' of the elliptic product S'' by Theorem 6 must be concyclic with the poles P_1, P_2 of S, also with the poles 0 and ∞ of S_1 and hence lie on a straight line through the origin. Since, moreover, the poles of S lie on a line through the origin and are separated by the origin, it follows geometrically that the poles of S'' are separated by the poles of S_1 and of S. We have now the following result.

THEOREM 7. *If the poles of two elliptic substitutions are concyclic and separate one another, the product of the substitutions is also elliptic, and its poles are concyclic with the two poles of each factor and are separated by them.*

We have now the equal negative segment-products, $OP_1'' \cdot OP_2'' = OP_1 \cdot OP_2$. Consider now *any* elliptic substitutions S and S'' whose two poles lie on a line through the origin, and on opposite sides in such a manner that the above segment products are equal. Then its poles P_1'', P_2'' are concyclic with P_1, P_2 and are separated by the same. Consequently the product SS'' will be elliptic, its two poles Q_1, Q_2 being concyclic with the poles of S, also of S'' and separated by the same. It follows at once by simple geometry that Q_1 and Q_2 must lie on some line through the origin and be so situated on opposite sides of the

origin that $OQ_1 \cdot OQ_2 = OP_1 \cdot OP_2 = OP_1'' \cdot OP_2''$. Hence the continued product of such elliptic substitutions with one another and with S_1 can give rise only to elliptic substitutions with the same properties. Thus the group generated by S and S_1 is composed only of elliptic substitutions.

Consider now any third non-loxodromic substitution which combines with our two elliptic substitutions S and S_1 so as to generate a non-loxodromic group. If this third substitution is hyperbolic, it cannot have a pole common with either S or S_1 since their product would then be loxodromic (Section 1). But if the hyperbolic substitution has no pole common with either elliptic substitution, then by Theorem 2 the poles of the two elliptic substitutions must be pairs of inverse points with respect to a common circle. They therefore lie on an orthogonal circle without separating one another, which contradicts our hypothesis regarding S and S_1. If, on the other hand, the third substitution is parabolic, its pole cannot be a pole of both elliptic substitutions. Then, by Theorem 5, there must be a hyperbolic substitution in the group generated, a possibility which has just been excluded. It follows that the third substitution must be elliptic. If elliptic, its poles P_1', P_2' must be concyclic with the poles of S, also with the poles of S_1 and hence lying on a straight line through the origin. Then, like the poles of S, they must lie on opposite sides of the origin and be so situated that $OP_1' \cdot OP_2' = OP_1 \cdot OP_2$. The considerations of the preceding paragraph then apply. We thus arrive at the following theorem.



THEOREM 8. *If the poles of two given elliptic substitutions are concyclic and separate one another, any other non-loxodromic substitution will combine with them so as to generate a non-loxodromic group if, and only if, it is an elliptic substitution whose poles are concyclic with the two poles of each of the two given substitutions. Any number of such substitutions will combine with one another and with the two given substitutions so as to give an elliptic group, and the poles of any two substitutions of the group are concyclic and separate one another.*

By forming the product of the two values of z in (4) we find

$$\overline{OP}_1 \cdot \overline{OP}_2 = |b|/|c|,$$

the value of which was seen to be the same for all substitutions of our elliptic group containing S_1. Since also $b \neq 0$ and $c \neq 0$, it is possible by a change of variable $w = mz$ to make simultaneously $|b| = |c|$ in all substitutions of our elliptic group. The change does not affect the values of a and d nor

destroy the unimodular character of the substitutions. Since in each substitution a and d are conjugate imaginaries and $bc = ad - 1 < 0$, we may place

$$a = e + fi, \quad d = e - fi, \quad b = g + hi, \quad -c = g - hi.$$

The substitutions of the group then take the canonical form

$$z' = \frac{(e + fi)z + g + hi}{-(g - hi)z + e - fi}, \tag{12}$$

where $e^2 + f^2 + g^2 + h^2 = 1$. This is the familiar form for the group of rotations of a sphere.

There remains for consideration the case of two elliptic substitutions (1) and (2) whose poles lie on a straight line through the origin without separating one another. These two substitutions leave invariant a circle around the origin, having a radius whose square is equal to $\overline{OP}_1 \cdot \overline{OP}_2 = |b|/|c|$. This is therefore invariant for the group which they generate. We might now proceed in a manner parallel to that for the case of two elliptic substitutions whose poles separate one another, but instead of doing so we shall establish that the group generated contains a hyperbolic substitution. For consider the product obtained by following the substitution

$$z' = \frac{dk^{-1}z - bk}{-ck^{-1}z + ak} \qquad (S^{-1}S_1^{-1})$$

with the substitution (3). This has the form

$$z' = \frac{(ad - bck^{-2})z + ab(1 - k^2)}{cd(1 - k^{-2})z + (ad - bck^2)}, \qquad (SS_1S^{-1}S^{-1}). \tag{13}$$

Here

$$(ad - bck^{-2}) + (ad - bck^2) = 2 - bc(k - k^{-1})^2 > 2,$$

inasmuch as bc is positive and $k - k^{-1}$ is a pure imaginary because $k = e^{i\phi}$. Accordingly (13) is a hyperbolic substitution, the poles of which lie upon the above invariant circle.

The poles of our two elliptic substitutions and of (13) fulfill the requirement of Theorem 2, and we have therefore the following result.

THEOREM 9. *When the poles of two elliptic substitutions lie upon a common circle without separating one another, they generate a non-loxodromic group in which hyperbolic substitutions are included.*

An immediate corollary of this theorem is that if a group contains only a finite number of substitutions (necessarily elliptic), it must be a group of elliptic substitutions in which the poles of any pair are concyclic and separate one another. The group can therefore be transformed into the familiar form (12) of the group of rotations of a sphere.

Another corollary is that the modular group* and, more generally, any group with a "*principal circle*" whose interior is transformed into itself by every substitution of the group, cannot contain any subgroups with a finite number of substitutions other than cyclical subgroups of elliptical substitutions. For, every such subgroup must be composed of elliptical substitutions whose points are symmetrically situated with respect to this circle. Now the poles of two such substitutions will not separate one another on their common circle; hence, unless all the elliptic substitutions all have their poles in common the group must contain a hyperbolic substitution and therefore an infinite number of substitutions. But a finite group of elliptic substitutions with common poles make a cyclic subgroup.

The non-loxodromic group generated by two elliptic substitutions whose poles do not separate one another can be transformed into a form corresponding to (12). The invariant circle of (1) and (2) is one with its center at the origin for which

$$R^2 = \overline{OP}_1 \cdot \overline{OP}_2 = |b|/|c| .$$

Since the value of $\overline{OP}_1 \cdot \overline{OP}_2$ is the same for all substitutions of the group, we may as before effect a change of variable $w = mz$ so as to make simultaneously $|b| = |c|$ for each substitution of the group. Since now a and d are conjugate imaginaries while $bc > 0$, the group takes the form

$$z' = \frac{(e+fi)z + g + hi}{(g-hi)z + e - fi}, \tag{14}$$

where $e^2 + f^2 - g^2 - h^2 = 1$. This formula may be regarded as a second canonical form for a non-loxodromic group of substitutions without a common pole which include a hyperbolic substitution. When the invariant or "principal" circle is transformed into the real axis, the group takes the real unimodular form which has been previously noted.

5. Non-loxodromic groups. Interpretation of a group with principal circle upon a hyperboloid of two sheets

In Section 1 we enumerated the kinds of non-loxodromic groups with a pole common to all the substitutions of the group. The possibilities for a group without a common pole were exhausted in the preceding paragraphs. We obtained

6. An elliptic group, which could be transformed into the canonical form of a rotation group.

* Theorem 9 brings out the intrinsic reason for the fact which does not appear in the special proof given by Fricke; cf. *Vorlesungen ueber die Theorie der elliptischen Modulfunctionen*, vol. 1, p. 309, footnote.

Vivanti, l.c., pp. 161–3, shows that when two elliptic substitutions belong to a finite group, their poles separate one another.

All other possibilities are included under groups of substitutions which admit a common invariant circle. When this circle is transformed into the real axis, the substitutions take real unimodular form, and when into the unit circle, the form (14). It has been shown that unless all substitutions of the group have a common pole, a hyperbolic substitution is included. We have therefore the following four

Varieties of non-loxodromic groups with principal circle. (*No common pole.*)

7. A mixed elliptic, parabolic, hyperbolic group.

8. An elliptic-hyperbolic group.

An example of such a group is obtained by using as its generators unimodular substitutions (14) for which $f = h = 0$ and substitutions for which $e = g = 0$. The former are the hyperbolic substitutions of form (14) which have the poles ± 1, the latter are elliptic substitutions of period 2 which exchange the poles of the former. The combination of these substitutions give rise only to such substitutions. They are, in fact, the real substitutions of form (14).

9. A purely hyperbolic group.

10. A parabolic-hyperbolic group.

An example of a parabolic-hyperbolic group is afforded by the set of all unimodular substitutions (14) for which f, g, h are even integers and e is an odd integer. The inverse of such a substitution is of like character. In the product of two substitutions the four constants have the values

$$E = ee' - ff' + gg' + hh', \qquad G = eg' + ge' + fh' - hf',$$
$$F = ef' + fe' + gh' - hg', \qquad H = eh' + he' + gf' - g'f,$$

and accordingly E is odd and F, G, H are even.

It is interesting to see the relation of this group to the modular group

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad (\alpha\delta - \beta\gamma = 1),$$

in which $\alpha, \beta, \gamma, \delta$ are real integers. Let the latter group be transformed through the substitution

$$z' = \frac{\frac{1}{\sqrt{2}}(w + i)}{\frac{1}{\sqrt{2}}(iw + 1)}$$

which converts the real axis of z into the unit circle of the w-plane. Thereby the modular group is transformed into a group of form (14), in which

$$e = (\alpha + \delta)/2, \qquad f = (\beta - \gamma)/2, \qquad g = (\beta + \gamma)/2, \qquad h = (\alpha - \delta)/2.$$

It thus appears that the substitutions (14) with integral values of e, f, g, h

are a subgroup of the transformed modular group, and the particular parabolic-hyperbolic group under consideration is a sub-group of this subgroup.

The groups 7–10, taken in form (14), have an interpretation on the hyperboloid of two sheets, $\zeta^2 - \xi^2 - \eta^2 = 1$, closely analogous to the interpretation of the elliptic group (12) as a rotational group of the sphere. Put

$$x = \frac{\xi}{\zeta - 1}, \qquad y = \frac{\eta}{\zeta - 1}, \qquad z = x + iy = \frac{\xi + i\eta}{\zeta - 1}$$

and accordingly

$$\frac{\zeta + 1}{\zeta - 1} = x^2 + y^2.$$

To every point in the z-plane there is made to correspond by these equations a point on the hyperboloid, and conversely. The unit circle of the z-plane is transformed into the circular section of the hyperboloid at infinity, and the groups (14) are therefore groups of motion of the hyperboloid into itself which leave the circular section at infinity unaltered. Further, to any circle (or straight-line) of the z-plane there corresponds a plane section of the hyperboloid, and conversely. Since any linear transformation of the z-plane converts circles into circles, it follows that the motion of the hyperboloid must be a collineation into itself. A hyperbolic substitution (14) leaves invariant the circles passing through its poles on the unit circle, and therefore the corresponding invariant sections of the hyperboloid are hyperbolas cut out by parallel planes passing through two invariant points of its circular section at infinity. When the substitution (14) is parabolic, the two invariant points come together, and the invariant sections along which the points of the hyperboloid move are parabolas cut out by parallel planes. Lastly, when the substitution (14) is elliptic, it is easy to see analytically that the invariant sections are ellipses cut out by a system of parallel planes.

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ON A GENERAL CLASS OF INTEGRALS OF THE FORM

$$\int_0^\infty \phi(t)\, g(x+t)\, dt^*$$

BY

R. D. CARMICHAEL

INTRODUCTION

In previous memoirs† I have considered the general theory of a class of infinite series of the form

$$\Omega(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)} \tag{1}$$

where the coefficients $c_0, c_1, c_2, \cdots$ are independent of x and where the function $g(x)$ is single-valued in a sector V formed by two rays from zero to infinity and containing in its interior the positive axis of reals, is analytic at each point in this region for which $|x|$ is sufficiently large but finite, and has the asymptotic property expressed in the formula

$$g(x) \sim x^{P(x)}\, e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right). \tag{2}$$

Here $P(x)$ and $Q(x)$ are polynomials which we write in the form

$$P(x) = \mu_0 + \mu_1 x + \mu_2 x^2 + \cdots + \mu_k x^k \qquad (\mu_k \neq 0 \text{ if } k > 0),$$

$$Q(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_m x^m \qquad (\alpha_m \neq 0 \text{ if } m > 0).$$

(By r^s we mean $e^{s \log r}$, where the principal determination of $\log r$ is taken.) In case $k = 0$ we assume that $m > 1$ in order to avoid a case which is unimportant so far as our present objects are concerned.

By means of these functions $g(x)$ we now introduce the general class of integrals

$$J(x) = \int_0^\infty \phi(t) \frac{g(x+t)}{g(x)}\, dt, \tag{3}$$

* Presented to the Society, April 13, 1918.

† These memoirs will be referred to by the numbers in the following list: I. These Transactions, vol. 17 (1916), pp. 207–232. II. Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 407–425. III. American Journal of Mathematics, vol. 39 (1917), pp. 385–403. IV. American Journal of Mathematics, vol. 40 (1918), pp. 113–126.

where $\phi(t)$ is finite and single-valued for finite $t \geqq 0$, $|\phi(t)|$ has an upper bound in every finite interval (0τ), and $\phi(t)$ is integrable in every such interval.* It is obvious that deep-lying analogies exist between series $\Omega(x)$ and integrals $J(x)$.

The most important case of integrals $J(x)$ which has figured in the literature is that for which we have

$$g(x) = e^{-\frac{1}{2}x^2}, \qquad \phi(t) = \psi(t)e^{\frac{1}{2}t^2};$$

this gives rise to the Euler-Laplace integral†

$$I(x) = \int_0^\infty \psi(t)e^{-tx}\,dt, \tag{4}$$

the integral at the basis of the Laplace transformation. It is also a fundamental integral in Borel's integral definition of the sum of a divergent series.

Other special cases of particularly simple properties are suggested at once by the instances of series $\Omega(x)$ exhibited in Memoir II. One may consider also a generalization of integrals $J(x)$ analogous to the generalization $T(x)$ of the series $\Omega(x)$ given in Memoir II.

A sum of two particular integrals $J(x)$ also gives rise to an integral of interest. For the first of these integrals $J(x)$ we take

$$\phi(t) = \tfrac{1}{2}[u(t) - iv(t)]e^{-\frac{1}{2}i(t^2-t)}, \qquad g(x) = e^{-\frac{1}{2}ix(x-1)}, \qquad i = \sqrt{-1},$$

where $u(t)$ and $v(t)$ are real-valued functions of the real variable t and x is now to be restricted to real values. For the second of these we take functions ϕ and g obtained from the foregoing on replacing i by $-i$. Adding the two corresponding integrals $J(x)$ we obtain the integral

$$\int_0^\infty \{u(t)\cos tx + v(t)\sin tx\}\,dt. \tag{5}$$

This will be recognized as an integral having close analogies with Fourier series.‡

The object of this paper is to develop the first fundamental convergence properties of the general integrals $J(x)$. In Section 1 I show that the region of convergence of $J(x)$ is a half-plane and determine the orientation of the line which bounds this half-plane. In Section 2 the corresponding results

* All integrals involved in the paper are taken in the sense of Cauchy-Riemann.

† This integral has been treated by many writers. For the principal references up to 1912 see *Encyclopédie des Sciences Mathématiques*, II, 5, pp. 35–41. Among later papers may be mentioned the following: Pincherle, Acta Mathematica, vol. 36 (1913), pp. 269–280; Horn, Crelle's Journal, vol. 144 (1914), pp. 167–189, and vol. 146 (1915), pp. 95–115; Horn, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 24 (1916), pp. 309–329, and vol. 25 (1916), pp. 74–83.

‡ In a similar way a Fourier series may be obtained as a sum of two series $\Omega(x)$.

for the region of absolute convergence are stated. The boundary of the half-plane of absolute convergence has the same orientation as the boundary of the half-plane of convergence. In Section 3 I consider uniform convergence in closed regions and the consequent properties as to analyticity of functions defined by integrals $J(x)$. Finally, in Section 4 I determine the position of the lines of convergence and absolute convergence in terms of ϕ and g. The formulæ are analogous to those by which the radius of convergence of a power series is determined in terms of the coefficients. Throughout the paper I have freely used the methods by which Landau‡ has treated a special case of the integrals $J(x)$.

1. Character of the region of convergence of $J(x)$

A point ρ in the complex x-plane corresponding to which there exists a non-negative real value t_1 of t such that the function $g(x+t_1)/g(x)$ of x has a singularity at $x=\rho$ will be called an *exceptional point* for the integral $J(x)$. All other points will be called non-exceptional points. We shall also speak of $x=\rho$ as an *exceptional* or *non-exceptional number* or *value* in the respective cases.

Let x_0 and x_1 be two values of x which are non-exceptional for the integral $J(x)$ and suppose that $J(x_0)$ converges. We seek conditions under which $J(x_1)$ certainly converges.

Let us write

$$G(\tau)=\int_\alpha^\tau \phi(t)g(x_0+t)\,dt$$

where the non-negative real constant α is chosen so that $g(x_0+t)$ is different from zero for every $t \geqq \alpha$ and where $\tau \geqq \alpha$. Since $J(x_0)$ converges it is clear that $G(\tau)$ approaches a finite limit as τ becomes infinite. Consider the integral

$$(6)\quad U(\tau)=\int_\alpha^\tau \phi(t)g(x_1+t)\,dt=\int_\alpha^\tau \phi(t)g(x_0+t)\frac{g(x_1+t)}{g(x_0+t)}\,dt.$$

To prove that $J(x_1)$ converges it is clearly sufficient to show that $U(\tau)$ approaches a finite limit as τ becomes infinite. Integrating by parts in (6) we have the following formula (which we shall need later):

$$(7)\qquad U(\tau)=G(\tau)\frac{g(x_1+\tau)}{g(x_0+\tau)}-\int_\alpha^\tau G(t)\left(\frac{d}{dt}\frac{g(x_1+t)}{g(x_0+t)}\right)dt.$$

This brings us to a consideration of the asymptotic character of $g(\xi+t)$ with respect to t for the fixed value ξ of x. It is easy to see that we have a

‡ Sitzungsberichte der mathematisch-physikalischen Klasse der Königlichen Bayerischen Akademie der Wissenschaften zu München, vol. 36 (1906), pp. 151–218. See especially pp. 208–218.

relation of the form

$$g(\xi+t) \sim e^{p(t,\xi)\log\tau + q(t,\xi)}\left(1+\frac{c_1}{t}+\frac{c_2}{t^2}+\cdots\right) \tag{8}$$

where the c's are independent of t and where $p(t,\xi)$ and $q(t,\xi)$ are quantities which may be written in the form

$$\begin{aligned} p(t,\xi) &= \nu_0(t)+\nu_1(t)\xi+\nu_2(t)\xi^2+\cdots+\nu_k(t)\xi^k, \\ q(t,\xi) &= \beta_0(t)+\beta_1(t)\xi+\beta_2(t)\xi^2+\cdots+\beta_m(t)\xi^m \\ &\quad +\gamma_1(t)\xi+\gamma_2(t)\xi^2+\cdots+\gamma_k(t)\xi^k. \end{aligned} \tag{9}$$

Here the quantities ν, β, γ have the values

$$\nu_s(t) = \sum_{i=s}^{k}\binom{i}{s}\mu_i t^{i-s}, \qquad s=0,1,\cdots,k;$$

$$\beta_s(t) = \sum_{i=s}^{m}\binom{i}{s}\alpha_i t^{i-s}, \qquad s=0,1,\cdots,m; \tag{10}$$

$$\gamma_s(t) = \sum_{i=0}^{k}\mu_i t^{i-s}\left[\binom{i}{s-1}-\frac{1}{2}\binom{i}{s-2}+\frac{1}{3}\binom{i}{s-3}-\cdots\right], \qquad s=1,2,\cdots,k.$$

From (8) it follows that we have with respect to t the asymptotic relation

$$\frac{g(x_1+t)}{g(x_0+t)} \sim e^{\log t\{p(t,x_1)-p(t,x_0)\}+\{q(t,x_1)-q(t,x_0)\}}\left(1+\frac{d_1}{t}+\frac{d_2}{t^2}+\cdots\right), \tag{11}$$

where the d's are independent of t. We see that $p(t,x_1)-p(t,x_0)$ is a polynomial in t of degree $k-1$ and that $q(t,x_1)-q(t,x_0)$ is a polynomial in t and that its degree is $m-1$ in case m is greater than k. (If k is zero the corresponding difference function is zero.)

By differentiating we obtain from (11) the asymptotic relation*

$$\frac{d}{dt}\frac{g(x_1+t)}{g(x_0+t)} \sim e^{r(t)}\left(-\frac{d_1}{t^2}-\frac{2d_2}{t^3}-\cdots\right)+e^{r(t)}r'(t)\left(1+\frac{d_1}{t}+\cdots\right), \tag{12}$$

where $r(t)$ denotes the exponent on e in (11) and $r'(t)=dr/dt$, the relation (12) being valid in the interior of V.

From (7) and (12) we see that

$$U(\tau) = G(\tau)\frac{g(x_1+\tau)}{g(x_0+\tau)} - \int_a^\tau G(t)\,T(t)\,dt - \int_a^\tau G(t)\,S(t)\,dt, \tag{13}$$

where $T(t)$ and $S(t)$ are suitable functions asymptotic to the first and second terms, respectively, in the second member of (12).

* For a theorem justifying this differentiation see a paper by J. F. Ritt in Bulletin of the American Mathematical Society, vol. 24 (1918), pp. 225–227.

It is convenient to make a separation of cases. We suppose first that $k < m$. Then $m \geqq 2$, since by hypothesis $m > 1$ if $k = 0$. The dominant term in the exponent of the second member of (11), when this exponent is expressed as a polynomial in t and $\log t$, is

$$m\alpha_m (x_1 - x_0) t^{m-1}.$$

Now a constant A exists such that $|G(t)| < A$ for every real value of t not less than α. Hence if $R\{\alpha_m (x_1 - x_0)\}$* is negative, it is clear that each term in the second member of (13) approaches a finite limit as τ becomes infinite; and thence that $J(x_1)$ is convergent.

In the next place suppose that $k \geqq m$ and $k > 1$. Precisely similar considerations lead to the conclusion that $J(x_1)$ is convergent in case $R\{\mu_k (x_1 - x_0)\}$ is negative.

There remains the case in which $k = 1$ and $m = 0$ or 1. We shall show in this case that $J(x_1)$ is convergent provided that $R\{\mu_1 (x_1 - x_0)\}$ is negative. For this special case (8) takes the form

$$g(\xi + t) \sim t^{\mu_0 + \mu_1(\xi+t)} e^{\alpha_0 + \mu_1 \xi + \beta(\xi+t)} \left(1 + \frac{c_1}{t} + \cdots\right),$$

where $\beta = \alpha_1$ or 0 according as $m = 1$ or 0. Then

$$\frac{g(x_1 + t)}{g(x_0 + t)} \sim t^{\mu_1(x_1 - x_0)} e^{(\mu_1 + \beta)(x_1 - x_0)} \left(1 + \frac{d_1}{t} + \cdots\right). \tag{14}$$

From this we see readily that the integrals

$$\int_a^\infty G(t) T(t)\, dt, \qquad \int_a^\infty G(t) S(t)\, dt \tag{15}$$

converge. Hence $U(\tau)$ approaches a finite limit as τ becomes infinite and $J(x_1)$ is therefore convergent.

Hence we have the following theorem (cf. Theorem I of Memoir I).

THEOREM I. *Let x_0 and x_1 be two values of x which are non-exceptional for the integral $J(x)$ and suppose that $J(x_0)$ converges. Then $J(x_1)$ also converges provided $R(\sigma x_1) < R(\sigma x_0)$, where σ denotes α_m or μ_k according as m is or is not greater than k.*

By a *region C of convergence* of the integral $J(x)$ we shall mean a region such that $J(x)$ converges for every non-exceptional value of x in the interior of C and diverges for every non-exceptional value of x exterior to C. In a similar way we define a *region Γ of absolute convergence.*

By means of Theorem I it is easy to determine the character of the region of convergence of $J(x)$. Compare the related argument in Memoir I, p. 214. We have the following result.

* We employ the symbol $R(z)$ to denote the real part of z.

THEOREM II. *There exists a real number* λ *such that the region of convergence of* $J(x)$ *is bounded by the straight line* $R(\sigma x) = \lambda$ *and lies on that side of this line for which* $R(\sigma x) < \lambda$.

This result may be compared with Theorem II of Memoir I. As to convergence on the boundary of the region of convergence one may establish properties similar to those given for $\Omega(x)$ in Memoir I, pp. 214–216, the method of treatment for $J(x)$ being analogous to that for $\Omega(x)$.

2. CHARACTER OF THE REGION OF ABSOLUTE CONVERGENCE

In view of (11) we have from (6), by means of an obvious argument, the following theorems.

THEOREM III. *If* x_0 *and* x_1 *are two values of* x *which are non-exceptional for* $J(x)$ *and if* $J(x_0)$ *converges absolutely, then* $J(x_1)$ *converges absolutely provided that* $R(\sigma x_1) < R(\sigma x_0)$.

A ready consequence of this is the following theorem.

THEOREM IV. *There exists a real number* μ *such that the region of absolute convergence of the integral* $J(x)$ *is bounded by the straight line* $R(\sigma x) = \mu$ *and lies on that side of this line for which* $R(\sigma x) < \mu$.

3. UNIFORM CONVERGENCE

We shall now prove the following theorem (compare Theorem III of Memoir I).

THEOREM V. *The integral* $J(x)$ *converges uniformly in any closed domain* D *which lies within its region of convergence and contains no point which is exceptional for* $J(x)$ *or is a limit point of points which are exceptional for* $J(x)$.

From the character of D as defined in the theorem it is clear that a non-exceptional point x_0 and a positive constant ϵ exist such that $J(x_0)$ converges and $R(\sigma x_0)$ is less than $R(\sigma x_1) - \epsilon$ for every x_1 in D. Let us now denote the integral $U(\tau)$ of (6) by $U(x_1, \tau)$ in order to put in evidence the variable x_1 which is to range over D. To prove the theorem it is sufficient to show that

$$\lim_{\tau=\infty} U(x_1, \tau)$$

exists uniformly as to x_1 ranging over D. The changes (mostly verbal in character) which may be made in the proof of Theorem I in order to reach this conclusion are now obvious. The argument need not be given in detail.

As an immediate consequence of Theorem V we have the following theorem.

THEOREM VI. *The integral* $J(x)$ *defines a function* $J(x)$ *of* x *which is analytic at every non-exceptional point which lies in the interior of its region of convergence and is not a limit point of exceptional points, and the derivatives of* $J(x)$ *at every such point may be found by differentiating the integral* $J(x)$ *under the sign of integration.*

4. Convergence numbers for the integral $J(x)$

For the determination of the convergence numbers we have the following theorem which we shall now demonstrate (compare Theorem XII and corollary of Memoir I).

Theorem VII. *The convergence [absolute convergence] number λ [μ] for the integral $J(x)$ is expressed in terms of ϕ and g by the following relations:*

(1) *in case $k < m$ we have*

$$\lambda = -\limsup_{u=\infty} \frac{\log\left|\int_u^{u+1} \phi(t)g(t)\,dt\right|}{mu^{m-1}} \left[\mu = -\limsup_{u=\infty} \frac{\log\int_u^{u+1} |\phi(t)g(t)|\,dt}{mu^{m-1}}\right];$$

(2) *in case $k \geqq m$ and $k > 1$ we have*

$$\lambda = -\limsup_{u=\infty} \frac{\log\left|\int_u^{u+1} \phi(t)g(t)\,dt\right|}{ku^{k-1}\log u} \left[\mu = -\limsup_{u=\infty} \frac{\log\int_u^{u+1} |\phi(t)g(t)|\,dt}{ku^{k-1}\log u}\right];$$

(3) *in case $k = 1$ and $m = 0$ or 1 we have*

$$\lambda = -\limsup_{u=\infty} \frac{\log\left|\int_{e^{E(u)}}^{e^u} \phi(t)g(t)\,dt\right|}{u} \left[\mu = -\limsup_{u=\infty} \frac{\log\int_{e^{E(u)}}^{e^u} |\phi(t)g(t)|\,dt}{u}\right],$$

where $E(u)$ denotes the integral part of u.

Let us first prove the formulas for μ. Taking up case (1), let us write

$$\limsup_{u=\infty} \frac{\log\int_u^{u+1} |\phi(t)g(t)|\,dt}{mu^{m-1}} = \zeta.$$

In the first place we take ζ to be finite. Then for every positive constant $\delta/2$ there exists a U such that

$$\int_u^{u+1} |\phi(t)g(t)|\,dt < e^{mu^{m-1}(\zeta+\delta/2)}$$

provided that $u \geqq U$. To show that $J(x_1)$ converges absolutely it is therefore sufficient to prove the convergence of the series

$$\sum_{n=0}^{\infty} M_n\, e^{m(U+n)^{m-1}(\zeta+\delta/2)},$$

where M_n is the least upper bound of the values of $|g(x_1+t)/g(t)|$ in the interval $U+n \leqq t \leqq U+n+1$. We take x_1 a non-exceptional point such that $R(\alpha_m x_1) = -\zeta - \delta$. Employing (11) with x_0 replaced by 0 we conclude readily that the foregoing infinite series is convergent. Hence $J(x_1)$

converges absolutely when x_1 is a non-exceptional point such that $R(\alpha_m x_1) < -\zeta$. Hence $\mu \not< -\zeta$.

To prove that $\mu \not> -\zeta$ we proceed as follows. For every positive constant $\delta/2$ there exists an infinite sequence of numbers $u_1, u_2, u_3, \cdots$ such that $u_{i+1} - u_i \geqq 1$ and

$$\int_u^{u+1} |\phi(t)g(t)|dt > e^{mu^{m-1}(\zeta-\delta/2)}$$

for $u = u_1, u_2, u_3, \cdots$. Now let x_1 be a non-exceptional point for $J(x)$ such that $R(\alpha_m x_1) = -\zeta + \delta$. Then the value of the integral

$$I_i = \int_{u_i}^{u_i+1} \left| \phi(t)g(t)\frac{g(x_1+t)}{g(t)} \right| dt$$

clearly does not approach zero as i increases indefinitely, as one sees by aid of relation (11); whence it follows that $J(x_1)$ is not absolutely convergent. Hence $\mu \not> -\zeta$.

In case ζ is infinite it may be shown that for every M and positive δ a set of numbers $u_1, u_2, u_3, \cdots$ exists such that $u_{i+1} - u_i \geqq 1$ and

$$\int_{u_i}^{u_i+1} |\phi(t)g(t)|dt > e^{mu_i^{m-1}(M+\delta)}.$$

Then if x_1 is a non-exceptional value for $J(x)$ such that $R(\alpha_m x_1) = -M$, it is clear that the value of the integral I_i does not approach zero as i increases. Hence when ζ is infinite the integral $J(x)$ converges absolutely for no non-exceptional value of x.

We conclude therefore to the validity of the formula for μ in case (1).

The determination of the value of μ in cases (2) and (3) is so closely parallel to that employed in case (1) that it is unnecessary to give a statement of it.

Proceeding to establish the formula for λ in case (1), let us write

$$\limsup_{u=\infty} \frac{\log \left| \int_u^{u+1} \phi(t)g(t)dt \right|}{mu^{m-1}} = \zeta.$$

First let ζ be finite. Then for every positive constant $\delta/2$ there exists a U such that

$$\left| \int_u^{u+1} \phi(t)g(t)dt \right| < e^{mu^{m-1}(\zeta+\delta/2)} \qquad \text{when} \qquad u \geqq U.$$

Let x_1 be a non-exceptional point such that $R(\alpha_m x_1) = -\zeta - \delta$ and consider the second member of (13) for τ becoming infinite, x_0 now having the value zero. Through use of (11) it is easy to see that its first term approaches a finite limit as τ increases indefinitely. That each of the other terms does

so also may be shown readily through use of (12). Hence $J(x_1)$ converges and $\lambda \not< -\zeta$.

To prove that $\lambda \not> -\zeta$ we let x_0 be a non-exceptional point for which $J(x_0)$ converges and prove that for every positive constant δ there exists a U such that

$$\left|\int_u^{u+1} \phi(t)g(t)\,dt\right| < e^{\alpha_m u^{-1}(\xi+\delta)} \qquad \text{for} \qquad u \geqq U, \tag{16}$$

where $\xi = -R(\alpha_m x_0)$.

If in the computation by which (13) was derived we write u and $u+1$ for α and τ, respectively, and take $x_1 = 0$, we have

$$\left|\int_u^{u+1} \phi(t)g(t)\,dt\right| \leqq \left|G(\tau)\frac{g(\tau)}{g(x_0+\tau)}\right]_u^{u+1}\Bigg| + \int_u^{u+1} |G(t)T(t)|\,dt + \int_u^{u+1} |G(t)S(t)|\,dt.$$

From the convergence of $J(x_0)$ it follows that a constant B exists such that $|G(t)| < B$ when $t \geqq \alpha$. Through (11) and (12) one now concludes readily to the truth of (16).

Hence the formula for λ is valid in case (1) if ζ is finite. Moreover, the argument just ended shows that ζ can be infinite only when $J(x)$ diverges for every non-exceptional value x_0. Hence the formula for λ is valid in case (1) when ζ is infinite.

Case (2) can be dealt with in the same manner as case (1).

In case (3) let us write

$$\lim_{n=\infty} u^{-1} \log \left|\int_{e^{E(u)}}^{e^u} \phi(t)g(t)\,dt\right| = \zeta.$$

Suppose first that ζ is finite. Then for every positive constant $\delta/2$ a constant U exists such that

$$\left|\int_{e^{E(u)}}^{e^u} \phi(t)g(t)\,dt\right| < e^{u(\zeta+\delta/2)} \qquad \text{when} \qquad u \geqq U.$$

Let x_1 be a non-exceptional point for $J(x)$ such that $R(\mu_1 x_1) = -\zeta - \delta$. We shall prove that $J(x_1)$ converges.

We shall employ relation (13) with x_0 replaced by 0, α being chosen so that $\alpha = e^k$ where k is a positive integer not less than $E(U)$ and so that $g(t)$ is different from zero for $t \geqq \alpha$. To prove the convergence of $J(x_1)$ it is sufficient to show that $\lim_{\tau=\infty} U(\tau)$ exists and is finite, $U(\tau)$ being defined as in (6).

We have

$$\begin{aligned} G(\tau) &= \int_a^\tau \phi(t)g(t)\,dt. \\ &= \sum_{l=1}^{l} \int_{e^{k+l-1}}^{e^{k+l}} \phi(t)g(t)\,dt + \int_{e^{k+l}}^{\tau} \phi(t)g(t)\,dt, \end{aligned}$$

where $k+l$ is the greatest integer such that $e^{k+l} < \tau$. Hence

$$|G(\tau)| < \sum_{i=1}^{l} e^{(k+i)(\zeta+\delta/2)} + e^{\log\tau\cdot(\zeta+\delta/2)} < \log\tau\cdot\tau^{\zeta+\delta/2}.$$

Since $R(\mu_1 x_1) = -\zeta - \delta$ it is now easy to see from (13) by aid of (14) and (12) that $\lim_{\tau=\infty} U(\tau)$ exists and is finite.

From this it follows that $J(x_1)$ converges. Hence we conclude that $\lambda \not< -\zeta$ when ζ is finite.

Let us suppose next that x_0 is a non-exceptional point for which $J(x_0)$ converges. To complete the proof for λ in case (3) when ζ is finite it is sufficient to show that for every positive constant δ there exists a U such that

$$\left|\int_{e^{E(u)}}^{e^u} \phi(t)\,g(t)\,dt\right| < e^{u(\xi+\delta)} \qquad \text{for} \qquad u \geqq U, \tag{17}$$

where $\xi = -R(\mu_1 x_0)$.

If in the computation by which (13) was derived we write $e^{E(u)}$ for α, e^u for τ, and take $x_1 = 0$, we have

$$\left|\int_{e^{E(u)}}^{e^u} \phi(t)\,g(t)\,dt\right| \leqq \left|G(\tau)\frac{g(\tau)}{g(x_0+\tau)}\right]_{\tau=e^{E(u)}}^{\tau=e^u}\right| + \int_{e^{E(u)}}^{e^u} |G(t)\,T(t)|\,dt + \int_{e^{E(u)}}^{e^u} |G(t)\,S(t)|\,dt.$$

Since $J(x_0)$ converges it is clear that a constant B exists such that $|G(t)| < B$. Hence, from the last preceding inequality, by aid of (14) and (12), we see readily that (17) is a valid relation. Hence $\lambda \not> -\zeta$ when ζ is finite.

The argument just ended shows that ζ can be infinite only when $J(x)$ diverges for every non-exceptional value x_0. Hence the formula for λ in case (3) is valid whether ζ is finite or infinite.

This completes the proof of the theorem.

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TRANSFORMATIONS OF SURFACES APPLICABLE TO A QUADRIC*

BY

LUTHER PFAHLER EISENHART

If a conjugate system of curves, or *net*, N, on a surface S and a congruence G of straight lines are so related that the developables of G meet S in N, the net and congruence are said to be *conjugate*. Two nets conjugate to the same congruence are said to be in the relation of a *transformation* T, if the nets are not parallel. In a previous paper† the author developed a general theory of these transformations T. When two surfaces, S and $\bar{S}$, are applicable, there is a unique net on S which is deformed into a net on $\bar{S}$. Let N and $\bar{N}$ denote these nets. Peterson‡ showed that if a net N' parallel to N is known, a net $\bar{N}'$ parallel to $\bar{N}$ can be found by quadratures such that N' and $\bar{N}'$ are applicable. In a former paper§ the author showed that when two such parallel nets N' and $\bar{N}'$ are known, two new applicable nets N_1 and $\bar{N}_1$ can be found by a quadrature such that N_1 and $\bar{N}_1$ are T transforms of N and $\bar{N}$ respectively.

In the present paper these general results are applied to the case where $\bar{N}$ is a net on a quadric, that is when N lies upon a surface applicable to a quadric. We consider first the case where the quadric is a general central quadric not of revolution and find readily the T transforms of N into nets of the same kind as described in the following theorems:

THEOREM *A*. *If N is a net applicable to a central quadric, not of revolution, Q, there can be found by the solution of a Riccati equation and quadratures three sets of ∞^2 T transforms N_1 which are applicable to Q; these transforms are conjugate to ∞^1 congruences G, there being ∞^1 transforms conjugate to each congruence G; the lines of these congruences G through a point of N form a quadric cone; the tangent planes at points of a line of G to the nets N_1 conjugate to it envelope a quadric cone and the points on Q corresponding to these points of the nets N_1 on a line of G lie on a conic.*

THEOREM *B*. *If N is a net applicable to Q, there exist an infinity of sets*

* Presented to the Society, September 3, 1919.

† These Transactions, vol. 18 (1917), pp. 97–124. This paper will be referred to as M_1.

‡ *Ueber Curven und Flächen*, Moskau and Leipzig, 1868, p. 106.

§ These Transactions, vol. 19 (1918), pp. 167–185. This paper will be referred to as M_2.

of ∞^2 T transforms N_1 of N; these transforms are conjugate to ∞^2 congruences G; their determination requires the solution of a certain completely integrable system of eight equations.

The existence of these transformations was established in a different manner, and at great length, by Guichard in his Mémoire sur la deformation des quadrics* awarded one half of the Grand Prize of the French Academy of Sciences in 1909. However, his method did not reveal the relations between the nets $\overline{N}$ and $\overline{N}_1$ on the quadric to which N and the nets N_1 are applicable; consequently the property mentioned in the last part of Theorem A is new. Furthermore, Guichard did not show that the nets N_1 in Theorem B are T transforms of N. This fact is of especial interest because of the relation between these transformations and the *transformations* B_k of surfaces applicable to a quadric discovered by Bianchi,† awarded the other half of the Grand Prize. Subsequently Bianchi showed‡ that if N_1 and N_1' are two suitable B_k transforms of a net N applicable to a quadric, then N_1 and N_1' are in relation of a transformation T.

The transformations of Guichard will be referred to as transformations G_k, k being a constant which is the same for all the transformations of a set. Three particular values of k determine the three sets of transformations described in Theorem A. In Section 5 we establish the following theorem of permutability of these transformations:

THEOREM C. *If N_1 and N_2 are transforms of a net N applicable to a net $\overline{N}$ on a central quadric Q by means of transformations G_{k_1} and G_{k_2}, where $k_2 \neq k_1$, there can be found without quadratures a net N_{12} applicable to Q which is in the relation of transformations G'_{k_2} and G'_{k_1} with N_1 and N_2 respectively.*

In certain cases it is possible also to apply this theorem when $k_2 = k_1$.

Guichard limited his consideration to the case where Q is a central quadric not of revolution. In Section 6 we consider the case of nets applicable to nets on a central quadric of revolution. We find that Theorem B holds equally true for this case, and that Theorem A applies with the exception that there is only one set of these transformations. However, we find that in addition such a net admits two sets of ∞^2 transformations T for which the congruences are normal. We consider also the case where Q is a sphere, real or imaginary, and discover that Theorem B holds in this case, that N and N_1 are composed of lines of curvature on surfaces of constant gaussian curvature, and that they are the transformations established by Bianchi.§

In Section 7 we establish transformations T of nets applicable to a paraboloid P. When P is not a surface of revolution, Theorems A and B apply, with the

* Mémoires à l'Académie, ser. 2, vol. 34 (1909).

† Ibid.

‡ Rendiconti dei Lincei, ser. 5, vol. 20 (1911), p. 145.

§ *Lezioni di geometria differenziale*, second edition, Chapter 20.

difference that there are only two sets of transformations of the former kind. When P is a surface of revolution, Theorem B applies but not Theorem A. Also these are transformations with normal congruences analogous to those of a net applicable to a net on a central quadric of revolution.

1. Transformations T of applicable nets

If N is a net, the cartesian coördinates, x, y, z, of the net satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = A\frac{\partial\theta}{\partial u} + B\frac{\partial\theta}{\partial v}. \tag{1}$$

A net N' is said to be parallel to N when its tangents are parallel to the corresponding tangents to N. If x', y', z' are the coördinates of N', then

$$\begin{aligned} &\frac{\partial x'}{\partial u} = h\frac{\partial x}{\partial u}, \qquad \frac{\partial y'}{\partial u} = h\frac{\partial y}{\partial u}, \qquad \frac{\partial z'}{\partial u} = h\frac{\partial z}{\partial u}, \\ &\frac{\partial x'}{\partial v} = l\frac{\partial v}{\partial x}, \qquad \frac{\partial y'}{\partial v} = l\frac{\partial y}{\partial v}, \qquad \frac{\partial z'}{\partial v} = l\frac{\partial z}{\partial v}, \end{aligned} \tag{2}$$

where h and l are a pair of solutions of

$$\frac{\partial h}{\partial v} = (l-h)A, \qquad \frac{\partial l}{\partial u} = (h-l)B. \tag{3}$$

Conversely, every pair of solutions of (3) determines a parallel net N'.

We call equation (1) the *point equation of* N. The coördinates of N' satisfy a similar equation. Moreover, to each solution θ of (1) there corresponds a solution θ' of the point equation of N'. It is determined by

$$\frac{\partial\theta'}{\partial u} = h\frac{\partial\theta}{\partial u}, \qquad \frac{\partial\theta'}{\partial v} = l\frac{\partial\theta}{\partial v}. \tag{4}$$

If θ and θ' are any pair of corresponding solutions of the point equations of N and N', the functions x_1, y_1, z_1, defined by equations of the form

$$x_1 = x - \frac{\theta}{\theta'}x', \tag{5}$$

are the cartesian coördinates of a net N_1 which is a T transform of N.* Conversely, the most general transformation T is defined in this way.

By means of the above formulas we establish the following relations:

$$x_1 - \frac{\theta}{\theta'\dfrac{\partial}{\partial u}\left(\dfrac{\theta}{\theta'}\right)}\frac{\partial x_1}{\partial u} = x - \frac{\theta}{\dfrac{\partial\theta}{\partial u}}\frac{\partial x}{\partial u},$$

$$x_1 - \frac{\theta}{\theta'\dfrac{\partial}{\partial v}\left(\dfrac{\theta}{\theta'}\right)}\frac{\partial x_1}{\partial v} = x - \frac{\theta}{\dfrac{\partial\theta}{\partial v}}\frac{\partial x}{\partial v}.$$

* M_1, p. 109.

Consequently the tangents to the curves $v = \text{const.}$ or $u = \text{const.}$ at corresponding points of N and N_1 meet in the points, P_1, P_2, whose coördinates are of the above forms. If we take another transformation T given by (5) with θ replaced by another solution θ_1 of (1), we find that the tangent plane to the transform meets the corresponding tangent planes to N and N_1 in the point whose coördinates are of the form

$$\xi = x - \frac{\left(\theta\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial\theta}{\partial v}\right)\frac{\partial x}{\partial u} - \left(\theta\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial\theta}{\partial u}\right)\frac{\partial x}{\partial v}}{\frac{\partial\theta}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta}{\partial v}\frac{\partial\theta_1}{\partial u}}.$$

From the form of this expression it follows that this point lies also on the tangent plane to the transform of N whose coördinates are of the form obtained by replacing θ in (5) by $\theta + c\theta_1$, where c is any constant, and θ' by a solution of the corresponding equations (4). Hence the corresponding tangent planes of the transforms obtained by varying c envelop a cone.

In order that corresponding points, M and M_1, of N and N_1 be equidistant from the corresponding point P_1, and also from P_2, defined above, we must have

$$\sum x'^2\frac{\partial\theta}{\partial u} = 2\theta'\sum x'\frac{\partial x}{\partial u}, \qquad \sum x'^2\frac{\partial\theta}{\partial v} = 2\theta'\sum x'\frac{\partial x}{\partial v}.$$

In consequence of (2) and (4) these equations are equivalent to

$$\theta' = c\sum x'^2,$$

where c is a constant. In this case the spheres with centers at P_1 and P_2 passing through M and M_1 intersect in a circle C orthogonal to N and N_1. Hence the circles C form a cyclic system,* and consequently N and N_1 lie on the sheets of the envelope of a two-parameter family of spheres.† Moreover, since $\sum x'^2$ is a solution of the point equation of N', the nets N', N, and N_1 consist of the lines of curvature of the surfaces on which they lie. Hence N and N_1 are all in the relation of a transformation of Ribaucour.‡

If $\bar{N}$ is a net applicable to N, its coördinates, $\bar{x}$, $\bar{y}$, $\bar{z}$, satisfy (1). Moreover, the net $\bar{N}'$, whose coördinates are given by quadratures of the form

$$\frac{\partial\bar{x}'}{\partial u} = h\frac{\partial\bar{x}}{\partial u}, \qquad \frac{\partial\bar{x}'}{\partial v} = l\frac{\partial\bar{x}}{\partial v}, \tag{6}$$

is parallel to $\bar{N}$ and is applicable to N'. Furthermore, the net $\bar{N}_1$ whose

* E. p. 428. A reference of this sort is to the author's *Differential Geometry*.

† E. p. 444, Ex. 14.

‡ These Transactions, vol. 17 (1916), pp. 437–458.

coördinates $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ are defined by

$$\bar{x}_1 = \bar{x} - \frac{\theta}{\theta'}\bar{x}' \tag{7}$$

is a T transform of $\bar{N}$.*

The common point equation of N' and $\bar{N}'$ admits the solution

$$\theta' = k(\sum \bar{x}'^2 - \sum x'^2), \tag{8}$$

where k is a constant, the symbol $\sum$ indicating the sum of three terms obtained from the three corresponding coördinates. We have shown† that for this value of θ' and the corresponding function θ, given by (4), the nets N_1 and $\bar{N}_1$ are applicable.

2. Transformations T of nets on a quadric.

Consider a net N on the general quadric, Q, whose equation is

$$ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fzx + 2rx + 2sy + 2tz + w = 0. \tag{9}$$

Since the coördinates are solutions of an equation of the form (1), we have on differentiating (9) with respect to u and v

$$\begin{aligned} a\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + c\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + d\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) \\ + e\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + f\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) = 0. \end{aligned} \tag{10}$$

Any net $N'(x')$ parallel to N is given by equations of the form (2). Consequently we have an equation of the form (10) in which x, y, z are replaced by x', y', z'. From this it follows that the function

$$\theta' = ax'^2 + by'^2 + cz'^2 + 2dx'y' + 2ey'z' + 2fz'x' \tag{11}$$

is a solution of the point equation of N'.‡ It is readily found that θ' and θ, given by

$$\begin{aligned} \theta = 2[axx' + byy' + czz' + d(xy' + yx') + e(yz' + zy') \\ + f(zx' + xz') + rx' + sy' + tz'], \end{aligned} \tag{12}$$

satisfy equations (4).

When these values are substituted in (5), it is found that the T transform $N_1(x_1)$ lies on Q. It can be shown that any congruence conjugate to a net N can be obtained by drawing through points of N lines whose direction-

* M_2. p. 170.

† M_2. p. 170.

‡ The function $\theta' \neq 0$, since N' cannot lie on a cone.

parameters are the coördinates of some net parallel to N. Hence we have the theorem of Ribaucour:*

Any congruence conjugate to a net on a quadric meets the quadric again in a net to which it is conjugate.

Let $N'(x')$ and $N''(x'')$ be two nets parallel to a net N, and $N_1(x_1)$ and $N_2(x_2)$ the T transforms of N determined by the pairs of corresponding functions θ_1, θ_1' and θ_2, θ_2'', where θ_1 and θ_2 are two solutions of (1). In place of (4) we have

$$(13)\quad \frac{\partial\theta_1'}{\partial u} = h_1\frac{\partial\theta_1}{\partial u}, \qquad \frac{\partial\theta_1'}{\partial v} = l_1\frac{\partial\theta_1}{\partial v}, \qquad \frac{\partial\theta_2''}{\partial u} = h_2\frac{\partial\theta_2}{\partial u}, \qquad \frac{\partial\theta_2''}{\partial v} = l_2\frac{\partial\theta_2}{\partial v},$$

where h_1, l_1 and h_2, l_2 are pairs of solutions of (3). In addition there are functions θ_1'' and θ_2', determined to within additive constants by the quadratures

$$(14)\quad \frac{\partial\theta_1''}{\partial u} = h_2\frac{\partial\theta_1}{\partial u}, \qquad \frac{\partial\theta_1''}{\partial v} = l_2\frac{\partial\theta_1}{\partial v}, \qquad \frac{\partial\theta_2'}{\partial u} = h_1\frac{\partial\theta_2}{\partial u}, \qquad \frac{\partial\theta_2'}{\partial v} = l_1\frac{\partial\theta_2}{\partial v},$$

since θ_1 and θ_2 are solutions of (1). We have shown† that a net N_1''' of coordinates x_1''', y_1''', z_1''' is given by equations of the form

$$(15)\quad x_1''' = x'' - \frac{\theta_1''}{\theta_1'}x',$$

and that N_1''' is parallel to N_1; also that the functions

$$(16)\quad \theta_{12} = \theta_2 - \frac{\theta_1}{\theta_1'}\theta_2', \qquad \theta_{12}''' = \theta_2'' - \frac{\theta_1''}{\theta_1'}\theta_2'$$

are corresponding solutions of the point equations of N_1 and N_1'''. Moreover, we have shown also that the functions x_{12}, y_{12}, z_{12}, of the form

$$(17)\quad x_{12} = x_1 - \frac{\theta_{12}}{\theta_{12}'''}x_1''',$$

are the coördinates of a net N_{12} which is a T transform of N_1 and also of N_2. Since θ_1'' and θ_2' are determined only to within arbitrary additive constants, there are accordingly ∞^2 such nets N_{12}.

We apply these results true for any net to the particular case when N is on the quadric Q, and also N_1 and N_2, that is when θ_1 and θ_2 are of the form (12). In order that θ_{12}''' and θ_{12} be of the form (11) and (12) with x', y', z'; x, y, z replaced by x_1''', y_1''', z_1'''; x_1, y_1, z_1 respectively, we must have

$$(18)\quad \begin{aligned}\theta_1'' + \theta_2' - 2[\,&ax'x'' + by'y'' + cz'z'' + d(x'y'' + x''y')\\ &+ e(y'z'' + y''z') + f(z'x'' + z''x')] = 0.\end{aligned}$$

*Comptes Rendus, vol. 74 (1872), p. 1491.

† M_1, p. 111.

By differentiation it is found that the left-hand member of this equation is constant, and consequently the additive constants in θ_1'' and θ_2' can be chosen in ∞^1 ways so that (18) shall hold. Hence

If N_1 and N_2 are T transforms of N and all three nets lie on $Q(9)$, there are ∞^1 other nets N_{12} on Q which are T transforms of N_1 and N_2; they can be found by a quadrature.

3. Transformations T of nets C on the deforms of a central quadric

We say that a net N is C when it admits an applicable net $\bar{N}(\bar{x})$. We consider in particular the case when $\bar{N}$ lies on the central quadric Q,

$$a\bar{x}^2 + b\bar{y}^2 + c\bar{z}^2 = 1, \tag{19}$$

where no two of the coefficients a, b, c are equal.

Let $N'(x')$ and $\bar{N}'(\bar{x}')$ be a pair of applicable nets parallel to N and $\bar{N}$ respectively, the coördinates being given by (2) and (6). We have seen that N_1 and $\bar{N}_1$ defined by (5) and (7) are applicable, if θ' is given by (8). Moreover, from (11) it follows that $\bar{N}_1$ will lie on Q, if

$$a\bar{x}'^2 + b\bar{y}'^2 + c\bar{z}'^2 = k(\sum \bar{x}'^2 - \sum x'^2). \tag{20}$$

There are two cases to consider, according as k is equal to a, b, or c or when k is not equal to one of these constants.

If $k = a$, equation (20) may be written

$$x'^2 + y'^2 + z'^2 + t'^2 + w'^2 = 0, \tag{21}$$

where

$$t' = \sqrt{\frac{b}{a} - 1}\,\bar{y}', \qquad w' = \sqrt{\frac{c}{a} - 1}\,\bar{z}'. \tag{22}$$

As thus defined t' and w' are solutions of the common point equation of N' and $\bar{N}'$. The functions

$$t = \sqrt{\frac{b}{a} - 1}\,\bar{y}, \qquad w = \sqrt{\frac{c}{a} - 1}\,\bar{z} \tag{23}$$

are the corresponding solutions of the common point equation (1) of N and $\bar{N}$. Moreover, since $\sum \bar{x}^2 - \sum x^2$ is a solution of (1), so also is $x^2 + y^2 + z^2 + t^2 + w^2$, in consequence of (19). Consequently we have

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial t}{\partial u}\frac{\partial t}{\partial v} + \frac{\partial w}{\partial u}\frac{\partial w}{\partial v} = 0. \tag{24}$$

Hence x, y, z, t, w may be looked upon as the coördinates of an *orthogonal net* in 5-space, that is a generalization of a net composed of lines of curvature in ordinary space. Since an equation of the form (24) in $x', \cdots, w'$ is a conse-

quence of (21), it follows that our problem consists in finding parallel nets in 5-space satisfying (21). We consider this problem in the next section and show that there are ∞^1 such nets parallel to an orthogonal net in 5-space.

When N and $\bar{N}$ are known, we have by (23) an orthogonal net in 5-space. Suppose we have a parallel net satisfying (21). Then $\bar{y}'$ and $\bar{z}'$ follow directly from (22), and $\bar{x}'$ is obtained by the quadrature (6). Thus $\bar{x}'$ involves an arbitrary additive constant e. Consequently there result ∞^1 T transforms N_1 of N. The coördinates of N_1 are of the form

$$x_1 = x - \frac{2[a\bar{x}(\bar{x}' + e) + b\bar{y}\bar{y}' + c\bar{z}\bar{z}']}{a(\bar{x}' + e)^2 + b\bar{y}'^2 + c\bar{z}'^2} x'. \tag{25}$$

Corresponding points of these nets N_1 lie on the same line of the congruence G_1 conjugate to N of direction parameters x', y', z'. The function θ of this transformation is

$$\theta = 2(a\bar{x}\bar{x}' + b\bar{y}\bar{y}' + c\bar{z}\bar{z}') + 2ea\bar{x},$$

and hence from Section 1 it follows that the tangent planes at corresponding points of the nets $\bar{N}_1$ envelop a cone. If ξ_0, η_0, ζ_0 are the coördinates of the vertex, the equation of the tangent plane is of the form

$$(\xi - \xi_0)X_1 + (\eta - \eta_0)Y_1 + (\zeta - \zeta_0)Z_1 = 0,$$

where ξ, η, ζ are current coördinates, and X_1, Y_1, Z_1 are direction-parameters of the normal to N_1. When the expressions for the latter are calculated, it is found that they involve e to the second degree. Consequently the tangent planes envelop a quadric cone.

Since the direction-parameters of the congruences $\bar{G}$ of the transformations from $\bar{N}$ into $\bar{N}_1$ are $\bar{x}' + e$, $\bar{y}'$, $\bar{z}'$, it follows that corresponding points of the ∞^1 nets $\bar{N}_1$ lie on a conic, the section of Q by a plane parallel to the plane of lines from O to the points $(\bar{x}' + e, \bar{y}', \bar{z}')$ as e varies.

When $k = b$ or c, we get ∞^1 other transformations of the same kind, and Theorem A has been established, except for the statement concerning the congruences G, which will be proved in the next section.

If k in (20) is any constant other than a, b, c, or 0, this equation may be written

$$x'^2 + y'^2 + z'^2 + s'^2 + t'^2 + w'^2 = 0, \tag{26}$$

where

$$s' = \sqrt{\frac{a}{k} - 1}\,\bar{x}', \qquad t' = \sqrt{\frac{b}{k} - 1}\,\bar{y}', \qquad w' = \sqrt{\frac{c}{k} - 1}\,\bar{z}'. \tag{27}$$

Evidently s', t', w' are solutions of the common point equation of N' and $\bar{N}'$, and s, t, w, defined by

$$s = \sqrt{\frac{a}{k} - 1}\,\bar{x}, \qquad t = \sqrt{\frac{b}{k} - 1}\,\bar{y}, \qquad w = \sqrt{\frac{c}{k} - 1}\,\bar{z},$$

are solutions of the common point equation of N and $\bar{N}$, as is also the function $x^2+y^2+z^2+s^2+t^2+w^2$. Hence x, y, z, s, t, and w may be looked upon as the cartesian coördinates of an orthogonal net in 6-space, and our problem reduces to the determination of parallel nets in 6-space for which (26) holds. In the next section we show that there are ∞^2 such parallel nets. When a set of functions satisfying (26) is known, the functions $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ follow directly from (27). Hence there is only one transform N_1 of N for each set of solutions, and Theorem B has been proved.

4. Orthogonal nets in n-space

Let $x_1, \cdots, x_n$ be the cartesian coördinates of an orthogonal net in n-space. These coördinates are solutions of an equation of the form (1) such that

$$\sum_{i=1}^{n} \frac{\partial x_i}{\partial u}\frac{\partial x_i}{\partial v} = 0. \tag{28}$$

We put

$$\frac{\partial x_i}{\partial u} = \sqrt{E}\,\xi_i, \qquad \frac{\partial x_i}{\partial v} = \sqrt{G}\,\eta_i \qquad (i = 1, \cdots n), \tag{29}$$

the functions ξ_i and η_i being chosen so that

$$\sum \xi^2 = \sum \eta^2 = 1, \qquad \sum \xi\eta = 0.$$

In this case equation (1) can be given the form

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial \log \sqrt{E}}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{G}}{\partial u}\frac{\partial \theta}{\partial v}, \tag{30}$$

and we have

$$\frac{\partial \xi}{\partial v} = \frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}\eta, \qquad \frac{\partial \eta}{\partial u} = \frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}\xi.$$

By algebraic processes we can find $n(n-2)$ functions $Y_j^{(i)}$ $(i = 1, \cdots n;\ j = 1, \cdots n-2)$ such that

$$\Delta = \begin{vmatrix} Y_1^{(1)} & \cdots & Y_1^{(n)} \\ \cdot & \cdot & \cdot \\ Y_{n-1}^{(1)} & \cdots & Y_{n-1}^{(n)} \\ \xi_1 & \cdots & \xi_n \\ \eta_1 & \cdots & \eta_n \end{vmatrix}$$

is an orthogonal determinant. Since

$$\sum_{i=1}^{n} Y_j^{(i)}\xi_i = 0, \qquad \sum_{i=1}^{n} Y_j^{(i)}\eta_i = 0,$$

we have equations of the form

$$\frac{\partial Y_j^{(i)}}{\partial u} = \sum_k P_{jk} Y_k^{(i)} + A_j \xi_i,$$

$$\frac{\partial Y_j^{(i)}}{\partial v} = \sum_k Q_{jk} Y_k^{i} + B_j \eta_i,$$

$$\frac{\partial \xi_i}{\partial u} = -\sum_k A_k Y_k^{(i)} - m\eta_i, \qquad \frac{\partial \xi_i}{\partial v} = n\eta_i,$$

$$\frac{\partial \eta_i}{\partial u} = m\xi_i, \qquad \frac{\partial \eta_i}{\partial v} = -\sum_k B_k Y_k^{(i)} - n\xi_i,$$

where

$$m = \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v}, \qquad n = \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u}.$$

Expressing the condition of integrability of the first two of these equations, we find that the functions P, Q, A, and B must satisfy

$$\frac{\partial}{\partial v} P_{jl} - \frac{\partial}{\partial u} Q_{jl} = \sum_k Q_{jk} \cdot \sum_l P_{kl} - \sum_l Q_{kl} \cdot \sum_k P_{jk},$$

(31)

$$\frac{\partial B_j}{\partial u} = \sum_k P_{jk} B_k + A_j m, \qquad \frac{\partial A_j}{\partial v} = \sum_k Q_{jk} A_k + B_j n.$$

We wish to show that it is posisble to define a set of functions $y_i^{(k)}$, which are the elements of an orthgonal substitution of order $n-2$, so that the functions $X_k^{(i)}$, defined by

$$X_k^{(i)} = y_1^{(k)} Y_1^{(i)} + \cdots + y_{n-2}^{(k)} Y_{n-2}^{(i)} \quad \begin{pmatrix} i = 1, \cdots n, \\ k = 1, \cdots n-2 \end{pmatrix},$$

satisfy the equations of the form

$$\frac{\partial X_k^{(i)}}{\partial u} = a_k \xi_i, \qquad \frac{\partial X_k^{(i)}}{\partial v} = b_k \eta_i. \tag{32}$$

In fact, substituting in these equations and making use of the above results, we find that the y's must satisfy the equations

$$\frac{\partial y_i^{(k)}}{\partial u} + \sum_{j=1}^{n-2} y_j^{(k)} P_{jl} = 0, \qquad \frac{\partial y_i^{(k)}}{\partial v} + \sum_{j=1}^{n-2} y_j^{(k)} Q_{jl} = 0. \tag{33}$$

It is readily found that the conditions of integrability of this system are satisfied, in consequence of (31), which establishes the existence of the functions $X_k^{(i)}$ forming with the ξ's and η's the elements of an orthogonal determinant, such that (32) holds.* In addition to these equations we have also

$$\frac{\partial \xi_i}{\partial u} = -\sum_k a_k X_k^{(i)} - m\eta_i, \qquad \frac{\partial \xi_i}{\partial v} = n\eta_i,$$

(34)

$$\frac{\partial n_i}{\partial u} = m\xi_i, \qquad \frac{\partial \eta_i}{\partial v} = -\sum_k b_k X_k^{(i)} - n\xi_i.$$

* Guichard, Annales de l'École Normale, ser. 3, vol. 14 (1897), pp. 498–516.

The conditions of integrability of (32) and (34) are readily found to be

$$\frac{\partial a_k}{\partial v} = m b_k, \qquad \frac{\partial b_k}{\partial u} = n a_k,$$

$$\frac{\partial m}{\partial v} + \frac{\partial n}{\partial u} + \sum_k a_k b_k = 0. \tag{35}$$

In terms of these functions the coördinates of the given net may be given the form

$$x_i = \sum_{k=1}^{n-2} p_k X_k^{(i)} + q\xi_i + r\eta_i, \tag{36}$$

where p_k, q, and r are determinate functions.

When we express the condition that the functions x_i, given by (36), satisfy (29), we find that

$$\frac{\partial p_k}{\partial u} = a_k q, \qquad \frac{\partial p_k}{\partial v} = b_k r,$$

$$\frac{\partial q}{\partial u} = -\sum_k a_k p_k - mr + \sqrt{E}, \qquad \frac{\partial q}{\partial v} = nr, \tag{37}$$

$$\frac{\partial r}{\partial u} = mq, \qquad \frac{\partial r}{dv} = -\sum_k b_k p_k - nq + \sqrt{G}.$$

If we put

$$2\omega = \sum_{i=1}^{n} x_i^2, \tag{38}$$

we have

$$\frac{\partial \omega}{\partial u} = \sqrt{E}\, q, \qquad \frac{\partial \omega}{\partial v} = \sqrt{G}\, r. \tag{39}$$

From (28) it follows that ω is a solution of (30). Conversely, if ω' is any other solution of (30), the functions q' and r' given by

$$\frac{\partial \omega'}{\partial u} = \sqrt{E}\, q', \qquad \frac{\partial \omega'}{\partial v} = \sqrt{G}\, r', \tag{40}$$

satisfy the fourth and fifth of equations (37). These functions and the functions p_k' obtained by the quadratures

$$\frac{\partial p_k'}{\partial u} = a_k q', \qquad \frac{\partial p_k'}{\partial v} = b_k r',$$

determine an orthogonal net N', parallel to N, whose coördinates are of the form

$$x_i' = \sum_k p_k' X_k^{(i)} + q' \xi_i + r' \eta_i.$$

Moreover,

$$2\omega' = \sum x_i'^2 = \sum_k p_k'^2 + q'^2 + r'^2.$$

From these results it follows that for the case

$$\sum x_i'^2 = 0,$$

it is necessary and sufficient that q' and r' be zero, and that

$$x_i' = \sum_k e_k X_k^{(i)}, \qquad \sum e_k^2 = 0, \tag{41}$$

where the e's are constants.

In the particular case treated in the preceding section, where (21) was to be satisfied, we have that $k = 3$. Since the above equations are homogeneous, there are consequently only ∞^1 sets of functions x', y', z', t', w' satisfying (21). Moreover, when we eliminate the e's from $\sum e_k^2 = 0$ and the three expressions for x', y', and z', we get a homogeneous quadratic in x', y', and z', and consequently the lines of the transformations through a point of N form a quadric cone. Furthermore, the determination of the functions $y_i^{(k)}$ from (33) is the same problem as finding the direction-cosines of a trihedral in 3-space whose rotations are given. This requires the solution of a Riccati equation.*

5. Theorems of permutability of transformations G_k

Let $N_1(x_1)$ and $N_2(x_2)$ be transforms of a net N applicable to a net $\bar{N}$ on Q, and let $\bar{N}_1(\bar{x}_1)$ and $\bar{N}_2(\bar{x}_2)$ be the nets on Q to which N_1 and N_2 are applicable. Let θ_1 and θ_2 be the functions of these transformations, where

$$\theta_1' = k_1(\sum \bar{x}'^2 - \sum x'^2), \qquad \theta_2'' = k_2(\sum \bar{x}''^2 - \sum x''^2), \tag{42}$$

$\bar{x}'$, $\bar{x}''$; x', x'' being obtained from $\bar{x}$ and x by equations of the form (13). We seek under what conditions the net N_{12} whose coördinates are of the form (17) is applicable to a net $\bar{N}_{12}$ on Q.

If we put

$$\theta_{12}''' = k_2(\sum \bar{x}_1'''^2 - \sum x_1'''^2), \tag{43}$$

where $\bar{x}_1'''$ is given by an equation similar to (15), we find that this expression for θ_{12}''' is equal to that given by (16), if

$$k_2\,\theta_1' + k_1\,\theta_2' - 2k_1\,k_2(\sum \bar{x}''\,\bar{x}' - \sum x''\,x') = 0. \tag{44}$$

If the left-hand member of this equation is differentiated, it is found that it is constant, in consequence of (14) and (42). Hence the additive constants entering in θ_1'' and θ_2' can be chosen so that (44) holds. In order that $\bar{N}_{12}$

* E., p. 159.

given by an equation similar to (17) be on Q, we must have

$$\theta_1'' + \theta_2' = 2(a\bar{x}'\,\bar{x}'' + b\bar{y}'\,\bar{y}'' + c\bar{z}'\,\bar{z}''), \tag{45}$$

as follows from (18).

Consider first the case where $k_2 \neq k_1$. Solving (44) and (45) for θ_1'' and θ_2', we find expressions which satisfy (14) and (42). Hence Theorem C has been proved. This theorem is equally true when k_1 and k_2, or either is equal to a, b, or c.

In order that $k_2 = k_1$, it follows from (44), (45), and (27) that we must have

$$x'\,x'' + y'\,y'' + z'\,z'' + s'\,s'' + t'\,t'' + w'\,w'' = 0,$$

where from (41)

$$x' = \sum_j e_j' X_j, \qquad x'' = \sum_j e_j'' X_j, \qquad \cdots.$$

Consequently the constants e' and e'' must satisfy

$$\sum e_j'^2 = 0, \qquad \sum e_j''^2 = 0, \qquad \sum e_j' e_j'' = 0.$$

This is possible when j can take on the values 1, 2, 3, 4, that is when k is not equal to a, b, or c, and then it is possible in an infinity of ways. We say that two such transformations G_k are complementary. In this case θ_1'' or θ_2' must be found by a quadrature and the other follows from (45). Hence

If N_1 and N_2 are complementary G_k transforms of a net N, there are ∞^1 nets N_{12} which are G transforms of N_1 and N_2; they can be found by a quadrature.

6. When Q is a central quadric of revolution, or a sphere

When $N(x)$ is applicable to a net on the quadric of revolution Q,

$$ax^2 + b(y^2 + z^2) = 1, \tag{46}$$

the transformations G_k, as described in Theorem B, exist as in the case of the general quadric. However, there is only one set of transformations of the type described in Theorem A. They are G_a.

When in equation (20) we put $k = b$, the result may be written

$$x'^2 + y'^2 + z'^2 + s'^2 = 0, \tag{47}$$

where

$$s' = \sqrt{\frac{a}{b} - 1}\,\bar{x}'. \tag{48}$$

If we put

$$s = \sqrt{\frac{a}{b} - 1}\,\bar{x},$$

the functions x, y, z, s are the cartesian coördinates of an orthogonal net in

4-space, since $x^2 + y^2 + z^2 + s^2$ also is a solution of (1). In this case the functions $y_i^{(k)}$ of Section 4 may be taken in the form

$$y_1^{(1)} = \cos\phi, \qquad y_1^{(2)} = \sin\phi, \qquad y_2^{(1)} = -\sin\phi, \qquad y_2^{(2)} = \cos\phi.$$

Then the determination of ϕ from (33) becomes an algebraic problem. There are two sets of solutions satisfying (47). They are of the forms

$$x' = X_1^{(1)} + iX_2^{(1)}, \qquad x'' = X_1^{(1)} - iX_2^{(1)}.$$

In either case $\bar{x}'$ follows directly from (48), but $\bar{y}'$ and $\bar{z}'$ are given by quadratures of the form (6) in which h and l are known. Hence both $\bar{y}'$ and $\bar{z}'$ involve arbitrary additive constants. Therefore either congruence conjugate to N is conjugate to ∞^2 T transforms N_1, which are applicable to Q.

If we keep the additive constant of $\bar{y}'$ or $\bar{z}'$ fixed and allow the other to vary, we see as in Section 3 that the tangent planes to these ∞^1 nets N_1 at corresponding points envelop a quadric cone, and the corresponding points on Q lie on a conic.

Ribaucour has shown that if the point equation of a net N on a surface S of coördinates x, y, z admits a solution R such that $x^2 + y^2 + z^2 - R^2$ also is a solution, the spheres with centers on S and of radius R are enveloped by two surfaces Σ_1 and Σ_2 upon which the lines of curvature correspond to the net N on S, and consequently the lines joining points of S to corresponding points of Σ_1 and Σ_2 form two normal congruences conjugate to N, which are the only normal congruences conjugate to N other than the one normal to S. Moreover, it can be shown that each of these congruences has for direction-parameters the coördinates x', y', z' of a net N' parallel to N for which $x'^2 + y'^2 + z'^2 = R'^2$, where x', y', z', R' are the solutions of the point equation of N' corresponding to x, y, z, R respectively.* This is the situation in the case of the above transformations. In fact the congruences of these transformations are the congruences of normals to the sheets of the envelopes of the spheres with centers on N and of radius $R = \sqrt{1 - (a/b)}\,\bar{x}$. We remark also that the distances from a point $(\bar{x}, \bar{y}, \bar{z})$ of $\overline{N}$ on Q to the foci on the axis of revolution are $R \pm 1/\sqrt{a}$. Evidently, the congruences normal to the sheets of the envelope of the spheres with centers on N and of radius $R \pm 1/\sqrt{a}$ are the same as the preceding.

Beltrami† has shown that if the lines of a normal congruence are invariably bound to a surface S, the congruence is changed into a new normal congruence when S is deformed. In consequence of this theorem and the facts set forth in the preceding paragraph, if lines are drawn from points of $\overline{N}$ to the foci of Q on the axis of revolution, these lines become the two congruences G_1 and G_2

* M_1, p. 110.

† E. p. 403.

of the two transformations, when $\overline{N}$ is applied to N. Hence we have the

THEOREM. *Let N be a net applicable to a net $\overline{N}$ on a central quadric of revolution Q; the lines joining points of $\overline{N}$ to the foci of Q on the axis of revolution become lines of two normal congruences, G_1 and G_2, conjugate to N when $\overline{N}$ is applied to N; there can be found by two quadratures ∞^2 nets N_1 conjugate to G_1 and ∞^2 nets N_2 conjugate to G_2 all of which are applicable to ∞^2 nets $\overline{N}_1$ and ∞^2 nets $\overline{N}_2$ on Q; the nets N_1 or N_2 can be grouped into ∞^1 families of ∞^1 nets such that their tangent planes at points on the same line of the congruence form a quadric cone and the corresponding points of the applicable nets on Q lie on a conic.*

The transformations G_k and G_a of nets applicable to a net $\overline{N}$ on Q (46) admit the theorem of permutability C. In addition there is an analogous theorem of permutability when N_1 is a transform of N in accordance with the above theorem and N_2 is a transform by a G_a or a G_k, when $k \neq b$.

When Q is a sphere, real or imaginary, with the equation

$$a(\bar{x}^2 + \bar{y}^2 + \bar{z}^2) = 1,$$

an applicable net N consists of the lines of curvature of a surface of constant gaussian curvature, since every net on Q is orthogonal. If we take $k = a$, we have from (20)

$$x'^2 + y'^2 + z'^2 = 0,$$

which is impossible for 3-space according to Section 4. Consequently only general transformations G_k exist, and the processes of finding them are the same as in Sections 3, 4 for the general quadric. From (20) and (11) we have $\theta' = \sum x'^2 \cdot ak/(k - a)$. Hence from Section 1 it follows that N and N_1 are in the relation of a transformation of Ribaucour, and consequently these are the transformations of surfaces of constant gaussian curvature discussed by Bianchi.*

7. TRANSFORMATIONS T OF NETS APPLICABLE TO A PARABOLOID

If the equation of the paraboloid P is taken in the form

$$a\bar{x}^2 + b\bar{y}^2 + 2\bar{z} = 0, \tag{49}$$

equations (11) and (12) become

$$\theta' = a\bar{x}'^2 + b\bar{y}'^2, \qquad \theta = 2(a\bar{x}\bar{x}' + b\bar{y}\bar{y}' + \bar{z}').$$

In place of (20) we have

$$\sum x'^2 + \left(\frac{a}{k} - 1\right)\bar{x}'^2 + \left(\frac{b}{k} - 1\right)\bar{y}'^2 - \bar{z}'^2 = 0. \tag{50}$$

When k is equal to a or b, we have transformations of the type described

* L. c.

in Theorem A, with the difference that there are only two such groups of transformations of nets applicable to P. In like manner there are ∞^2 transformations G_k, for each value of k other than a, b, or zero. Moreover, these transformations admit a theorem of permutability analogous to Theorem C.

When $b = a$ in (49) and P is a surface of revolution, Theorem B holds true. There are no transformations as described by Theorem A. When we put $k = a$ in (50), it becomes

$$\sum x'^2 = \bar{z}'^2.$$

Proceeding as in Section 6, we find that there are two sets of ∞^2 transforms of N, the ∞^2 transforms of either set being conjugate to a congruence which is normal to one of the sheets of the envelope of spheres with centers on N and of radius $\bar{z}$. From a well-known property of the parabola and the above theorem of Beltrami it follows that these normal congruences are obtained by drawing from points of $\overline{N}$ lines to the focus of P and lines perpendicular to the tangent plane to P at its vertex, and then applying $\overline{N}$ to N. Hence we have for nets applicable to a paraboloid of revolution a theorem the same as for a central quadric of revolution (section 6) except with reference to the method of generating the normal congruences of the transformations.

PRINCETON UNIVERSITY

NOTE ON TWO THREE-DIMENSIONAL MANIFOLDS WITH THE SAME GROUP*

BY

J. W. ALEXANDER

1. Poincaré has proved that there exist 3-dimensional manifolds with identical Betti numbers and coefficients of torsion but which are nevertheless distinguishable in the sense of Analysis Situs by the fact that they have different groups.† It is proposed to set up an example of two 3-dimensional manifolds which are by no means equivalent but which cannot even be differentiated by their groups. Since the manifolds have the same group, they also have the same Betti numbers and coefficients of torsion, for in the 2-sided 3-dimensional case, these other invariants are functions of the group alone.

2. To clarify certain points that will arise later on in the discussion, we shall begin by recalling explicitly several well-known and quite obvious properties of an anchor ring, A, lying in ordinary 3-dimensional space.

(a) The ring A may be made simply connected by means of two cuts, one along a generating circle, the other along a circle which intersects the first in one and only one point. If, therefore, we denote by a and b respectively the operations of describing these two curves in preassigned senses, the group of the surface will be the group generated by the operations a and b connected by the single relation $aba^{-1}b^{-1} \equiv 1$. In the discussion that follows, the two circles determining a and b will also be referred to by the symbols for the corresponding group operations, when this can be done without confusion.

(b) The circle a bounds a 2-cell that reduces the interior of the anchor ring to a simply connected piece. Any other simple closed curve with this property may be deformed continuously on the surface of the ring into coincidence with a.‡

(c) A surface which in its initial position coincides with the anchor ring A

* Presented to the Society, September 4, 1919.

† Rendiconti di Palermo, vol. 18 (1904), p. 45. See also Dehn, Mathematische Annalen, vol. 69 (1910).

‡ Every curve on the anchor ring which bounds in the interior region corresponds to a group operation a^k, and if the curve is to be a simple curve, we must have $k = \pm 1$, cf. Poincaré, loc. cit., pp. 26 et seq.

may be deformed continuously within the ring until it reduces to an arbitrarily thin tube bounding as small a neighborhood as we please of some suitably chosen closed curve interior to the ring.

3. With these facts in mind, let us consider a manifold, M, made up of two 3-dimensional pieces separated by a common surface of genus 1 and such that either of the pieces may be mapped with its boundary on the interior and surface of an anchor ring A in ordinary 3-space. We cannot expect to crowd the entire manifold into the 3-space without cutting it up or forcing it to overlap itself, but we can represent it conveniently in schematic form by a *diagram* made by mapping one of the regions in the manner indicated above and tracing on the surface A of the ring the boundary l of a 2-cell that reduces the omitted region to a simply connected piece.* The curve l, which is unique except for continuous deformations of itself (§2*b*), determines without ambiguity the way in which the missing elements must be attached to the diagram in order to completely reconstruct the manifold. Starting with any non-bounding curve l on the anchor ring, we may at once build up a manifold of the type M.

The invariants of the manifold can at once be read from the diagram. Let the operation of describing the curve l in a properly chosen sense be $a^\sigma b^\tau$ ($\tau \geqq 0$) in terms of the fundamental operations a and b, defined in Section 2*a*. The group of the manifold is then the group of the surface A modified by the addition of the two identical relations $a \equiv 1$, and $a^\sigma b^\tau \equiv 1$, corresponding to the boundaries of the cells that reduce the two regions of the manifold to simply connected pieces. In other words, leaving out the case $\tau = 0$ with which we shall not be concerned, it is the group generated by a single operation b such that $b^\tau \equiv 1$. The Betti numbers (as defined by Poincaré) are $P_1 = P_2 = 1$, since every curve bounds when counted τ times, and there is one coefficient of torsion, τ, whenever τ is greater than one.

4. From now on, the discussion will be restricted to two particular diagrams, obtainable as follows:

Trace on the anchor ring A five non-intersecting circles, each equivalent to the circle b and so spaced that their points of intersection with the circle a divide the latter into five equal arcs. Then, cut the anchor ring along the circle a, and make a continuous deformation of the cut surface into itself such that the edge of the cut containing the initial points of the b curves remains fixed, while the edge containing the terminal points rotates through an angle $\frac{2}{5}\pi$. Lastly, re-join the two edges of the cut. The diagram D thus obtained consists of the anchor ring A and a curve $l = ab^5$.

The second diagram D' will be obtained in a similar way, the sole difference

* This is a simple example of Heegaard's diagrams. Cf. Heegaard, *Inaugural dissertation*, Copenhagen, 1898, or the French translation of the same, Bulletin de la Société Mathématique de France, vol. 44 (1916), p. 161.

being that the edge of the cut will be rotated through the angle $\frac{4}{5}\pi$, so that the curve finally obtained will be $l' = a^2 b^5$. It may be well to observe that all the diagrams obtained in the above manner except that the edge of the cut is rotated through an angle $(2\pi k + \frac{2}{5}\pi)$, where k is any integer or zero, represent the same manifold as D, since any one of these diagrams (including the interior of the anchor ring) can be mapped uniformly on D. In a like manner, the diagrams obtained by a rotation through an angle $(2\pi k + \frac{4}{5}\pi)$ all represent the same manifold as D'. There is, however, the following essential difference between diagrams of the two classes D and D' which we shall use later. In diagrams of the class D, the points of intersection of the two curves a and l are passed in the same cyclic order (12345), whether we describe one or other of the curves in the positive sense. In diagrams of the class D', if the points are passed in the order (12345) as we describe the curve a, they are passed in the order (13524) as we describe the curve l'.

5. Let M be the manifold determined by the diagram D. Then as an immediate consequence of Section 2*c*, we can suppose that the points of the diagram correspond to those of the *entire manifold excepting* the points of an arbitrarily thin tube enclosing some curve c of the omitted region. Furthermore, if Δ be any other diagram defining the same manifold, we can assume, by again applying Section 2*c*, that Δ represents *only* that portion of the manifold within an arbitrarily thin tube about some curve d which can always be so chosen as not to intersect the curve c. Therefore, we can suppose that the surfaces of the two diagrams correspond to non-intersecting surfaces of the manifold, and we can map the diagram Δ on the interior of the diagram D, in the form of a thin tube T winding about in the portion of 3-space interior to the anchor ring A bounding the diagram D.

We are now ready to show that no diagram Δ defining the manifold M can possibly be the diagram D'.

6. We observe that the first Betti number of the portion of 3-space occupied by the diagram D and bounded by the anchor ring A is 2, since all curves in this region are homologous to zero or to the non-bounding curve b. Suppose we remove from this region the interior of the tube T. Then, however the tube T may wind about, the first Betti number of the remaining region R is 3, and to each curve in the region, there can be associated a symbol

$$\kappa a' + \lambda b,$$

where a' corresponds to the curve on the tube T that bounds a 2-cell making the interior of the tube simply connected. A *necessary* condition that a curve of the region R bound a 2-cell is that

$$\kappa a' + \lambda b \sim 0$$

in the notation of Poincaré, or, if we use an algebraic notation in which a' and b are merely treated as linearly independent marks,

$$\kappa a' + \lambda b \equiv 0.$$

Now, the symbol for the curve $l = ab^5$ on the diagram D is

$$\theta a' + 5b$$

where $|\theta|$ denotes the number of times (algebraically), that the tube T winds around the anchor ring A, and where the sign of θ depends on the determination of the positive direction on the curve a'. Therefore, a necessary condition that a curve of the region R bound a 2-cell in the entire part of the manifold M exterior to the tube T is that its symbol satisfy a relation

$$\kappa a' + \lambda b + \alpha(\theta a' + 5b) \sim 0$$

or, in the other notation,

$$\kappa a' + \lambda b \equiv 0 \mod (\theta a' + 5b). \tag{1}$$

Suppose Δ were the diagram D' of Section 4. Then the curve l' of the diagram would have the symbol

$$(5k + 2)a' \pm 5\theta b.$$

For, on describing the curve l' in the positive sense, we would pass over the points where it met the curve a' in the cyclic order (13524), so that each of the five arcs of l' would wind $k + \frac{2}{5}$ times around the tube T. Moreover, since the curve l' would bound in the part of the manifold exterior to the tube T, we would be led, by (1), to the congruence

$$(5k + 2)a' \pm 5\theta b \equiv 0 \mod (\theta a' + 5b). \tag{2}$$

However, it is easy to see that the congruence (2) is false. For, if $\theta = 0$, we are led to $5k + 2 = 0$, which is false, while if $\theta \neq 0$, we are led to $\theta^2 = \pm (5k + 2)$ which implies

$$\theta^2 \equiv \pm 2 \mod 5,$$

whereas θ^2 is always congruent to 0 or ± 1, modulo 5.

Therefore, the diagram D' does not determine the same manifold as the diagram D.

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A GENERAL SYSTEM OF LINEAR EQUATIONS*

BY

A. J. PELL

By virtue of the Hellinger-Radon integral,† which reduces in one special case to the Hellinger integral and in another special case to the sum of squares, a system of linear equations of the form

$$(a)\qquad \begin{aligned} x_i &= \sum_k a_{ik}\, x_k + \sum_\beta \int_a^b \frac{da_i^{(\beta)}(t)\, du^{(\beta)}(t)}{da_0^{(\beta)}(t)}, \\ \Delta u^{(\alpha)}(s) &= \sum_k \Delta \bar{a}^{(\alpha)}(s)\, x_k + \sum_\beta \int_a^b \frac{d_t\, \Delta A^{(\alpha,\beta)}(s,t)\, du^{(\beta)}(t)}{da_0^{(\beta)}(t)}, \end{aligned}$$

which involves a finite or denumerably infinite number of unknowns $\{x_i\}$ and $\{u^{(\alpha)}(s)\}$, is contained in the single equation

$$(b)\qquad \mathfrak{U} = A[\mathfrak{U}],$$

where $\mathfrak{U}$ is a function on a composite range and A is a linear functional transformation We consider in this paper the solution of such equations, the corresponding non-homogeneous and adjoint equations, both when the functional transformation A is completely continuous but not necessarily symmetric, and when it is limited and symmetric. The method used by us is the construction of an orthogonal system of functions on a composite range, and by means of this orthogonal system the transformation of the equations above into the linear equations

$$z_i = \sum_k b_{ik}\, z_k$$

in denumerably infinitely many unknowns in a Hilbert Space. Then the results of Hilbert's fourth memoir are simply carried over into the more general situation.

In his memoirs‡ and lectures E. H. Moore has considered general systems of linear equations in *General Analysis*, which include for example the mixed linear integral equation. While a special case of (*a*) is the mixed linear

* Presented to the Society, December, 1917.

† Radon, J., *Absolut additive Mengenfunktionen*, Wiener Akademie Sitzungsberichte, vol. 122, Abt. 2a, Heft 6–10 (1913), pp. 1295–1438.

‡ E. H. Moore, *On the foundations of the theory of linear integral equations*, Bulletin of the American Mathematical Society, vol. 18 (1912), pp. 334–362.

integral equation, that theory is not contained in this paper, for the functional transformation A is not limited in the sense defined here.

Following the lines of Hilbert's theory, J. Radon* develops a theory for linear equations of the form (b) where $\mathfrak{U}$ is a function on a simple range. His theory is confined to the completely continuous case, and yields at the same time the Hilbert theory of linear equations in denumerably infinitely many unknowns, and the theory of Fredholm's linear integral equations.

Section 6 deals with the properties of orthogonality and closure of a general matrix $L = (l^{(\alpha,\beta)}(E, E'))$, which may consist of $(l_{ik}, p_i^{(\alpha)}(s), \pi_k^{(\beta)}(t), \Lambda^{(\alpha,\beta)}(s,t))$ and includes as special cases the matrices formed from the coefficients of orthogonal systems of linear forms and linear differential forms. In his lectures E. H. Moore has given similar results for matrices in *General Analysis*.

I wish to acknowledge some helpful suggestions in regard to the Hellinger-Radon integral from Mr. T. H. Hildebrandt.

I. Definitions

In a space of n dimensions let I be defined by $a_i \leqq c_i \leqq x_i < d_i < b_i$ and let $\mathfrak{E}$ be a class of sets E of points of I such that

(1) If E_1 and E_2 belong to $\mathfrak{E}$, the sum $E_1 + E_2$ and the product $E_1 E_2$ belong to $\mathfrak{E}$.

(2) If $E_1, \cdots, E_n, \cdots$ belong to $\mathfrak{E}$ and are mutually distinct, $\sum_n E_n$ belongs to $\mathfrak{E}$.

We consider real single-valued functions $f(E)$ on $\mathfrak{E}$.

A function $f(E)$ is *absolutely additive* if, when $E_1, \cdots, E_n, \cdots$ belong to $\mathfrak{E}$ and are mutually distinct

$$f(\sum_n E_n) = \sum_n f(E_n).$$

A function $f(E)$ is *monotonic* if $f(E) \geqq 0$.

A monotonic, absolutely additive function $f_0(E)$ is called a *basis function* of the absolutely additive function $f(E)$, if $f(E) = 0$ when $f_0(E) = 0$.

The range of the variables i, k, α, and β is that of a sequence, which may be finite in some cases.

Hellinger-Radon integral.† Let $f_0(E)$ be a basis function of the absolutely additive function $f(E)$. Divide E into a finite number of distinct sets E_1, $E_2, \cdots, E_n$, and form the sum

$$\sum_{i=1}^{n} \frac{(f(E_i))^2}{f_0(E_i)},$$

* Radon, l. c.

† Radon, l. c.

the term being defined to be zero when $f_0(E_i) = 0$; if this sum has a limit as n is increased and the diameter of E_i approaches zero, the limit is denoted by the Hellinger-Radon integral

$$\int_E \frac{(df)^2}{df_0},$$

and $f(E)$ is *integrable* $H(f_0)$.

If $f(E)$ and $f_1(E)$ are absolutely additive, and integrable $H(f_0)$ on E, the sum

$$\sum_{i=1}^{n} \frac{f(E_i) f_1(E_i)}{f_0(E_i)}$$

has a limit as n is increased and the diameter of E_i approaches zero, and the limit is denoted by

$$\int_E \frac{df\, df_1}{df_0}.$$

A system of absolutely additive functions $\{f_i(E)\}$, each integrable $H(f_0)$ on E, is *closed*, if there exists no absolutely additive function $f(E) \not\equiv 0$, integrable $H(f_0)$, such that

$$\int_E \frac{df\, df_i}{df_0} = 0 \qquad (i = 1, 2, \cdots).$$

We denote by $\mathfrak{F}$ a system of absolutely additive functions $\{f^{(a)}(E)\}$ on $\mathfrak{E}_a$ classes of sets of points I_a defined by $a_i^{(a)} \leqq c_i^{(a)} \leqq x_i < d_i^{(a)} < b_i^{(a)}$ in spaces of finite number of dimensions. If the Hellinger-Radon integrals of these functions with basis functions $f_0^{(a)}(E)$ exist, and the sum

$$\sum_a \int_{I_a} \frac{(df^{(a)})^2}{df_0^{(a)}}$$

converges, we denote it by

$$\int \frac{(d\mathfrak{F})^2}{d\mathfrak{F}_0},$$

and say that $\mathfrak{F}$ is *integrable* $H(\mathfrak{F}_0)$. It can easily be shown that if $\mathfrak{F}$ and $\mathfrak{F}_1$ are integrable $H(\mathfrak{F}_0)$ the sum

$$\sum_a \int_{I_a} \frac{df^{(a)}\, df^{(a)}}{df_0^{(a)}}$$

converges, and we denote it by

$$\int \frac{d\mathfrak{F}\, d\mathfrak{F}_1}{d\mathfrak{F}_0}.$$

2. Orthogonal systems with given basis functions

Given a monotonic absolutely additive function $f_0(E)$ on I, there exists a closed system of absolutely additive, linearly independent functions

$f_i(E)$, which have $f_0(E)$ as a basis function and are integrable $H(f_0)$. Divide I into two mutually distinct subsets $E_1^{(1)}$ and $E_2^{(1)}$; divide each one of these sets into two mutually distinct subsets $E_i^{(2)}$ $(i = 1, 2, 3, 4)$; continue this process in such a way that the diameter of $E_i^{(n)}$ approaches zero. Define a system of absolutely additive functions as follows

$$f_i^{(n)}(E) = \begin{cases} f_0(E) & E = E_i^{(n)} \text{ or any subset of } E_i^{(n)}, \\ 0 & E \text{ distinct from } E_i^{(n)}. \end{cases}$$

From the system $f_1^{(1)}, f_2^{(1)}, f_1^{(2)}, f_2^{(2)}, \cdots$, drop out all the functions which are zero or linearly dependent on a finite number of the preceding, and let $\{f_i(E)\}$ be the system thus obtained. Obviously the functions $f_i(E)$ are integrable $H(f_0)$. For any absolutely additive function $f(E)$ such that

$$\int_I \frac{df\, df_i}{df_0} = 0 \qquad (i = 1, 2, \cdots),$$

it follows that $f(E_i^{(n)}) = 0$, and hence

$$\int_I \frac{(df)^2}{df_0} = 0,$$

and $f(E) \equiv 0$, so that the system $\{f_i(E)\}$ is closed.

A system of absolutely additive functions $\{p_i(E)\}$, which are all integrable $H(p_0)$, forms a *normalized orthogonal system with the basis function* $p_0(E)$ if

$$\int_I \frac{dp_i\, dp_k}{dp_0} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}. \tag{1}$$

For every monotonic, absolutely additive function $p_0(E)$, *a closed normalized orthogonal system* $\{p_i(E)\}$, *with* $p_0(E)$ *as a basis function can be constructed.* Because, if $\{f_i(E)\}$ is a closed system of linearly independent absolutely additive functions with $p_0(E)$ as a basis function, constants c_{ik} can be determined by the usual procedure, so that $p_i(E) = \sum_{k=1}^{k=i} c_{ik} f_k(E)$ satisfy the conditions (1).

By the usual method we obtain, for any normalized orthogonal system $\{p_i(E)\}$, Bessel's inequality

$$\sum_i \left(\int_I \frac{dp_i\, df}{dp_0} \right)^2 \leqq \int_I \frac{(df)^2}{dp_0}, \tag{2}$$

for any function $f(E)$ integrable $H(p_0)$. A special case of (2) is

$$\sum_i (p_i(E))^2 \leqq p_0(E),$$

and hence the series

$$\sum_i p_i(E) \int_I \frac{dp_i\, df}{dp_0}$$

is uniformly convergent.

If the orthogonal system $\{p_i(E)\}$ is closed

$$(E) = \sum_i p\ (E) \int_I \frac{dp_i\, df}{dp_0},$$

also

$$\int_I \frac{df\, dg}{dp_0} = \sum_i \int_I \frac{dp_i\, df}{dp_0} \int_I \frac{dp_i\, dg}{dp_0},$$

for every $f(E)$ and $g(E)$ integrable $H(p_0)$. From this last equality it follows that

$$p_0(E_1 E_2) = \sum_i p_i(E_1)\, p_i(E_2). \tag{3}$$

Let $\{p_0^{(\alpha)}(E)\}$ be a system, finite or denumerably infinite, of monotonic, absolutely additive functions, and $\{\overline{p}_i^{(\alpha)}(E)\}$ closed normalized orthogonal systems with $p_0^{(\alpha)}(E)$ as basis functions. Let $L = (l_{ik})$ be an orthogonal matrix, that is,

$$\sum_j l_{ij}\, l_{kj} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases},$$

and

$$\sum_j l_{ji}\, l_{jk} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}.$$

Let $(l_{ik}^{(\alpha)})$ be matrices made up from the columns of L, so that each column of L occurs in one and only one of the $(l_{ik}^{(\alpha)})$. The functions defined by

$$p_i^{(\alpha)}(E) = \sum_k l_{ik}^{(\alpha)}\, \overline{p}_k^{(\alpha)}(E)$$

are absolutely additive, form a system $\{\mathfrak{P}_i\}$ integrable $H(\mathfrak{P}_0)$, and satisfy the following *orthogonal* relations

$$\int \frac{d\mathfrak{P}_i\, d\mathfrak{P}_k}{d\mathfrak{P}_0} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

$$\sum_i p_i^{(\alpha)}(E_1)\, p_i^{(\beta)}(E_2) = \begin{cases} p_0^{(\alpha)}(E_1 E_2) & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}, \tag{4}$$

or

$$\sum_i \left(\int \frac{d\mathfrak{F}\, d\mathfrak{P}_i}{d\mathfrak{P}_0} \right)^2 = \int \frac{(d\mathfrak{F})^2}{d\mathfrak{P}_0}.$$

For every sequence $\{x_i\}$ of finite norm, $\sum_i x_i\, \mathfrak{P}_i$ represents a function integrable $H(\mathfrak{P}_0)$, and

$$\int \frac{d\mathfrak{F}\, d\Sigma_i x_i\, \mathfrak{P}_i}{d\mathfrak{P}_0} = \sum_i x_i \int \frac{d\mathfrak{F}\, d\mathfrak{P}_i}{d\mathfrak{P}_0}. \tag{5}$$

3. Linear functional transformations and matrices

In connection with a given matrix A of absolutely additive functions $(a^{(\alpha,\beta)}(E, E'))$, we introduce the following notations. $\mathfrak{A}_E^\alpha$ denotes the system of functions $a^{(\alpha,\beta)}(E, E')$ for given α and E; $\mathfrak{A}_E'^\alpha$ denotes the system of functions $a^{(\beta,\alpha)}(E', E)$ for given α and E. If $\mathfrak{A}_E^\alpha$, $\mathfrak{A}_E'^\alpha$, $\mathfrak{F}$, and $\mathfrak{G}$ are integrable $H(\mathfrak{A}_0)$, where $\mathfrak{A}_0$ is a system of monotonic, absolutely additive functions, we define the following functional transformations and bilinear form

$$A[\mathfrak{F}] = A_E^\alpha[\mathfrak{F}] = \int \frac{d\mathfrak{A}_E^\alpha\, d\mathfrak{F}}{d\mathfrak{A}_0},$$

$$A'[\mathfrak{F}] = A_E'^\alpha[\mathfrak{F}] = \int \frac{d\mathfrak{A}_E'^\alpha\, d\mathfrak{F}}{d\mathfrak{A}_0}, \tag{6}$$

and if $A[\mathfrak{G}]$ is integrable $H(\mathfrak{A}_0)$,

$$A(\mathfrak{F}, \mathfrak{G}) = \int \frac{d\mathfrak{F}\, dA[\mathfrak{G}]}{d\mathfrak{A}_0}.$$

If $A = (a^{(\alpha,\beta)}(E, E'))$ and $C = (c^{(\alpha,\beta)}(E, E'))$ are two matrices such that $\mathfrak{A}_E^\alpha$ and $\mathfrak{C}_E'^\alpha$ are integrable $H(\mathfrak{A}_0)$, the product $D = AC$ denotes the matrix formed from the elements

$$d^{(\alpha,\beta)}(E, E') = \int \frac{d\mathfrak{A}_E^\alpha\, d\mathfrak{C}_E'^\beta}{d\mathfrak{A}_0}.$$

The matrix $\Delta(\mathfrak{P}_0)$ is defined by the elements

$$\delta^{(\alpha,\beta)}(E, E') = \begin{cases} p_0^{(\alpha)}(E, E') & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}.$$

Given a system $\mathfrak{A}_0$ of monotonic, absolutely additive functions $a_0^{\alpha}(E)$, a linear functional transformation $T[\mathfrak{F}]$ is *limited with respect to* $\mathfrak{A}_0$, if, (1) it transforms every absolutely additive function $\mathfrak{F}$ integrable $H(\mathfrak{A}_0)$ into an absolutely additive function $T[\mathfrak{F}]$ integrable $H(\mathfrak{A}_0)$: (2) there exists a constant M independent of $\mathfrak{F}$ such that

$$\int \frac{(dT[\mathfrak{F}])^2}{d\mathfrak{A}_0} \leqq M \int \frac{[d\mathfrak{F}]^2}{d\mathfrak{A}_0}.$$

It follows immediately from the second condition that for any $\mathfrak{F}_n$ and $\mathfrak{F}$ integrable $H(\mathfrak{A}_0)$ and such that

$$\lim_{n\to\infty} \int \frac{(d(\mathfrak{F}_n - \mathfrak{F}))^2}{d\mathfrak{A}_0} = 0,$$

$$\lim_{n\to\infty} \int \frac{(dT[\mathfrak{F}_n - \mathfrak{F}])^2}{d\mathfrak{A}_0} = 0$$

and

$$\lim_{n\to\infty} T[\mathfrak{F}_n] = T[\mathfrak{F}].$$

Let $\{\mathfrak{P}_i\}$ be a closed orthogonal system with basis function $\mathfrak{A}_0$, and let $\{c_i\}$ be any sequence of finite norm. If $T[\mathfrak{F}]$ is a limited linear functional transformation, and $\mathfrak{F} = \sum_i c_i \mathfrak{P}_i$ then

$$T[\mathfrak{F}] = \sum_i c_i T[\mathfrak{P}_i],$$

and hence $T[\mathfrak{P}_i]$ is of finite norm. In the usual way† it can be shown that if $\mathfrak{G}$ is integrable $H(\mathfrak{A}_0)$

$$\int \frac{d\mathfrak{G} dT[\mathfrak{F}]}{d\mathfrak{A}_0} = \sum_i c_i \int \frac{d\mathfrak{G}\, dT[\mathfrak{P}_i]}{d\mathfrak{A}_0}, \tag{5'}$$

and hence

$$\int \frac{d\mathfrak{G} dT[\mathfrak{P}_i]}{d\mathfrak{A}_0}$$

is of finite norm. An adjoint transformation $T^*[\mathfrak{F}]$ can be defined by

$$T^*[\mathfrak{F}] = \sum_i \mathfrak{P}_i \int \frac{d\mathfrak{F} dT[\mathfrak{P}_i]}{d\mathfrak{A}_0}.$$

A matrix A of absolutely additive functions $a^{(\alpha,\beta)}(E, E')$, integrable $H(\mathfrak{A}_0)$, is *limited with respect to* $\mathfrak{A}_0$, if the functional transformation $A[\mathfrak{F}]$, defined in (6), is limited with respect to $\mathfrak{A}_0$.

THEOREM I. *A functional transformation T which is linear and limited with respect to $\mathfrak{A}_0$, gives rise to a matrix A of absolutely additive functions $a^{(\alpha,\beta)}(E, E')$ integrable $H(\mathfrak{A}_0)$ for every α and E, such that*

$$T = A[\mathfrak{F}] = \int \frac{d\mathfrak{A}_E^\alpha d\mathfrak{F}}{d\mathfrak{A}_0},$$

$$A(\mathfrak{F}, \mathfrak{G}) = A'(\mathfrak{G}, \mathfrak{F}),$$

and

$$|A(\mathfrak{F}, \mathfrak{G})| \leqq M,$$

for $\mathfrak{F}$ and $\mathfrak{G}$ such that

$$\int \frac{(d\mathfrak{F})^2}{d\mathfrak{A}_0} \leqq 1,$$

and

$$\int \frac{(d\mathfrak{G})^2}{d\mathfrak{A}_0} \leqq 1.$$

Let $\{\mathfrak{P}_i\}$ be a closed orthogonal system with basis function $\mathfrak{A}_0$, then the matrix B defined by the elements

$$b_{ik} = \int \frac{d\mathfrak{P}_i\, dT[\mathfrak{P}_k]}{d\mathfrak{A}_0} \tag{7}$$

† Hellinger, E., *Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen*, Journal für die reine und angewandte Mathematik, 1907, p. 239.

is limited.* If $\{x_i\}$ is a sequence of finite norm, then, from (5), the linearity of T, property (2) of a limited functional transformation, and from (5′)

$$\sum_k b_{ik} x_k = \sum_k x_k \int \frac{d\mathfrak{P}_i dT[\mathfrak{P}_k]}{d\mathfrak{A}_0} = \int \frac{d\mathfrak{P}_i d\sum_k x_k T[\mathfrak{P}_k]}{d\mathfrak{A}_0} = \int \frac{d\mathfrak{P}_i dT[\sum_k x_k \mathfrak{P}_k]}{d\mathfrak{A}_0}.$$

But $\sum_k x_k \mathfrak{P}_k$ is integrable $H(\mathfrak{A}_0)$, and from the third relation of (4) it follows that $\sum_k b_{ik} x_k$ is of finite norm, and B is limited. The functions

$$a^{(\alpha,\beta)}(E, E') = \sum_{i,k} p_i^{(\alpha)}(E) b_{ik} p_k^{(\beta)}(E') \tag{8}$$

are absolutely additive in E and E', integrable $H(\mathfrak{A}_0)$ for every α and E, also for every β and E', and are such that

$$T[\mathfrak{F}] = A[\mathfrak{F}].$$

From the properties of the limited matrix B, the remainder of the theorem follows.

Let $\mathfrak{F}^{(n)}$, $\mathfrak{F}$, $\mathfrak{G}^{(n)}$, $\mathfrak{G}$ be integrable $H(\mathfrak{A}_0)$, $\int((d\mathfrak{F}^{(n)})^2/d\mathfrak{A}_0)$ and $\int((d\mathfrak{A}^{(n)})^2/d\mathfrak{A}_0)$ bounded in n, and $\lim_n \mathfrak{F}^{(n)} = \mathfrak{F}$, $\lim_n \mathfrak{G}^{(n)} = \mathfrak{G}$. The bilinear form $A(\mathfrak{F}, \mathfrak{A})$ is *completely continuous* if

$$\lim_{n\to\infty} A(\mathfrak{F}^{(n)}, \mathfrak{A}^{(n)}) = A(\mathfrak{F}, \mathfrak{G}).$$

If $A(\mathfrak{F}, \mathfrak{G})$ is completely continuous, the bilinear form $B(x, y)$ is completely continuous.

4. Systems of linear equations. Completely continuous case

Let A and B be limited matrices connected by the relations (7) and (8), where $\{\mathfrak{P}_i\}$ is a closed orthogonal system with basis function $\mathfrak{A}_0$. If $\{m_i\}$ is a sequence of finite norm, the system of functions $\mathfrak{U}$ defined by

$$\mathfrak{U} = \sum_k m_k \mathfrak{P}_k \tag{9}$$

is integrable $H(\mathfrak{A}_0)$, and the sequence $\{m_i\}$ is expressed in terms of $\mathfrak{U}$ by

$$m_i = \int \frac{d\mathfrak{P}_i d\mathfrak{U}}{d\mathfrak{A}_0}. \tag{10}$$

The relations (9) and (10) establish a one-to-one correspondence between the solutions $\mathfrak{U}$, integrable $H(\mathfrak{A}_0)$, of the system of linear functional equations

$$\mathfrak{U} = A[\mathfrak{U}], \tag{11}$$

* Hellinger and Toeplitz, *Theorie der unendlichen Matrizen*, § 10, Mathematische Annalen, 1910, vol. 69.

and the solutions $\{m_i\}$ of finite norm of the system of equations

$$m_i = \sum_k b_{ik} m_k \qquad (i = 1, 2, \cdots). \tag{12}$$

Multiply both sides of (12) by $\mathfrak{P}_i$, sum with respect to i, then in the right hand side substitute for m_i from (10), and we obtain (11). Similarly we may pass from (11) to (12).

There is a one-to-one correspondence between the solutions of the adjoint systems

$$\mathfrak{W} = A'[\mathfrak{W}] \tag{13}$$

and

$$n_i = \sum_k b_{ki} n_k; \tag{14}$$

also between the solutions of the corresponding non-homogeneous equations

$$\overline{\mathfrak{U}} = \mathfrak{F} + A[\overline{\mathfrak{U}}],$$

where $\mathfrak{F}$ is integrable $H(\mathfrak{A}_0)$, and

$$\overline{m}_i = c_i + \sum_k b_{ik} \overline{m}_k, \tag{16}$$

where $\{c_i\}$ is of finite norm, and $\mathfrak{F} = \Sigma_k c_k \mathfrak{P}_k$.

From the known theorems* concerning the solutions of (12), (14), and (16), we obtain the following theorem, if we understand by a solution, one which is integrable $H(\mathfrak{A}_0)$.

THEOREM 2. *Let the matrix* $A = (a^{(\alpha,\beta)}(E, E'))$ *be limited with respect to* $\mathfrak{A}_0$, *and the bilinear form* $A(\mathfrak{F}, \mathfrak{G})$ *be completely continuous. If the homogeneous equations* (11) *have no non-trivial solutions, the non-homogeneous equations* (15) *have one and only one solution for every* $\mathfrak{F}$ *integrable* $H(\mathfrak{A}_0)$. *The homogeneous equations* (11) *can have only a finite number of linearly independent non-trivial solutions. If the equation* (11) *has n linearly independent non-trivial solutions* $\mathfrak{U}_i(E)$, *the adjoint equation* (13) *has n linearly independent non-trivial solutions* $\mathfrak{W}_i(E)$, *and the non-homogeneous equation* (15) *has solutions when and only when.*

$$\int \frac{d\mathfrak{F} d\mathfrak{W}_i}{d\mathfrak{A}_0} = 0 \qquad (i = 1, 2, \cdots, n).$$

When I_α consist of points in a space of one dimension, $a_0^{(1)}$ is formed from a monotonic non-decreasing function, constant except for a finite or denumerably infinite set of points of discontinuity in I, and $a_0^{(\alpha)}(\alpha > 1)$ formed

* Hilbert, D., *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, pp. 164–170.

from continuous, monotonic non-decreasing functions,* the equation (11) takes the form

$$
\begin{aligned}
x_i &= \sum_k a_{ik} x_k + \sum_{\beta=2}^{\infty} \int_a^b \frac{da_i^{(1,\beta)}(t)\, du^{(\beta)}(t)}{da_0^{(\beta)}(t)}, \\
\Delta u^{(\alpha)}(s) &= \sum_k \Delta a_k^{(\alpha,1)}(s)\, x_k + \sum_{\beta=2}^{\infty} \int_a^b \frac{d_t\, \Delta a^{(\alpha,\beta)}(s,t)\, du^{(\beta)}(t)}{da_0^{(\beta)}(t)},
\end{aligned}
\tag{17}
$$

in which the unknowns are $\{x_i\}$ and $\{u^{(\alpha)}(s)\}$. As an illustration of (17) in which the form $A(\mathfrak{F}, \mathfrak{G})$ is completely continuous, we give the following. The range of α and β is that of a single element; of i and k, a finite number n of elements; $a_{ik} = 1$ for $i = k$, $a_{ik} = 0$ for $i \neq k$; $a_i^{(1,2)}(s)$ are continuous functions and have continuous first derivatives $K_i(s)$; $a_i^{(2,1)}(s)$ are continuous functions and have continuous first derivatives $L_i(s)$; $a^{(2,2)}(s, t)$ is continuous in s and t, and has continuous first and second partial derivatives, $\partial a^{(2,2)}/\partial t$ and $(\partial^2 a^{(2,2)}/\partial s \partial t) = K(s, t)$; $a_0^{(2)}(s) = s$. The equations (17) are in this case equivalent to

$$
\begin{aligned}
0 &= \int_a^b K_i(t)\, \phi(t)\, dt \qquad (i = 1, 2, \ldots, n), \\
\phi(s) &= \sum_{k=1}^{n} L_k(s)\, x_k + \int_a^b K(s, t)\, \phi(t)\, dt,
\end{aligned}
\tag{18}
$$

where $\{x_i\}$ and $\phi(s)$ are the unknowns. The adjoint system is

$$
\begin{aligned}
0 &= \int_a^b L_i(t)\, \psi(t)\, dt, \\
\psi(s) &= \sum_{k=1}^{n} K_k(s)\, y_k + \int_a^b K(t, s)\, \psi(t)\, dt,
\end{aligned}
\tag{19}
$$

where $\{y_i\}$ and $\psi(s)$ are the unknowns. The system of non-homogeneous equations corresponding to (18) has the form

$$
\begin{aligned}
0 &= c_i + \int_a^b K_i(t)\, \bar{\phi}(t)\, dt \qquad (i = 1, 2, \cdots, n), \\
\bar{\phi}(s) &= f(s) + \sum_{k=1}^{n} L_k(s)\, \bar{x}_k + \int_a^b K(s, t)\, \bar{\phi}(t)\, dt,
\end{aligned}
$$

and the necessary and sufficient conditions that it have solutions when (19) has solutions y_{ki}, $\psi_k(s)$, are

$$
\sum_{i=1}^{n} c_i\, y_{ki} + \int_a^b f(t)\, \psi_k(t)\, dt = 0 \qquad (k = 1, 2, \cdots, m).
$$

* Radon, J., pp. 1321–1322, and Hildebrandt, T. H., *On integrals related to and extensions of the Lebesgue integrals*, p. 197, Bulletin of the American Mathematical Society, January, 1918.

5. Symmetric systems of linear equations

A limited matrix A is symmetric if $a^{(\alpha,\beta)}(E, E') = a^{(\beta,\alpha)}(E', E)$, and then the matrix B defined by (7) is also symmetric. When B is symmetric and limited, there exists† a spectrum s, lying within a finite interval of the λ-axis which consists of real characteristic numbers λ_i^*, or of a continuous spectrum (a^*, b^*) or of both, for the equations

$$\begin{aligned} \lambda^* l_k^* &= \sum_j b_{kj} l_j^*, \\ \int_\Delta \lambda dp_k^* &= \sum_j b_{kj} \Delta p_j^*(\lambda), \end{aligned} \tag{20}$$

where Δ is any subinterval of (a^*, b^*). Let $\{L_i^*(x)\}$ be the characteristic linear forms corresponding to λ_i^* and $\{dP^{*(\alpha)}(\lambda; x)\}$ the characteristic linear differential forms; they form an orthogonal system with basis functions $\{p_0^{*\alpha}\}$, that is

$$\begin{aligned} (L_i^*, L_k^*) &= \begin{cases} 1 & i = k \\ 0 & i \neq k, \end{cases} \\ (\Delta_1 P^{*(\alpha)}, \Delta_2 P^{*(\beta)}) &= \begin{cases} \Delta_{12} p_0^{*(\alpha)} & \alpha = \beta \\ 0 & \alpha \neq \beta, \end{cases} \\ (L_i^*, \Delta P^{*(\alpha)}) &= 0, \end{aligned} \tag{21}$$

where Δ_1 and Δ_2 are any two subintervals of (a^*, b^*), and Δ_{12} is the subinterval common to them. If for some values of α we allow $r_i^{(\alpha)}(\lambda)$ to be discontinuous functions, constant except at λ_i^* where they have finite discontinuities and the corresponding basis functions $r_0^{(\alpha)}(\lambda)$ have unit discontinuities, the two equations (20) may be expressed by the one equation

$$\int_\Delta \lambda dr_i^{(\alpha)}(\lambda) = \sum_j b_{kj} \Delta r_j^{(\alpha)}(\lambda), \tag{22}$$

where Δ is any subinterval of the spectrum s; and the orthogonal relations (21) by

$$(\Delta_1 R^{(\alpha)}, \Delta_2 R^{(\beta)}) = \begin{cases} \Delta_{12} r_0^{(\alpha)} & \alpha = \beta \\ 0 & \alpha \neq \beta. \end{cases} \tag{23}$$

The relation

$$l^{(\alpha,\beta)}(\lambda, E) = \sum_i r_i^{(\alpha)}(\lambda) p_i^{(\beta)}(E)$$

establishes a one-to-one correspondence between the solutions of (22), and the solutions $l^{(\alpha,\beta)}(\lambda, E)$, integrable $H(\mathfrak{A}_0)$ for every α and λ, of

$$\int_\Delta \lambda d_\lambda \mathfrak{L}_\lambda^{(\alpha)} = A[\Delta \mathfrak{L}_\lambda^{(\alpha)}], \tag{24}$$

where Δ is any subinterval of s. For the characteristic numbers λ_i^*, the equation (24) becomes

$$\lambda_i \mathfrak{L}_i = A[\mathfrak{L}_i].$$

† Hilbert, D., l. c., pp. 109–174; and Hellinger, E., l. c., pp. 210–271.

The orthogonal properties (23) yield the following orthogonal properties for $\mathfrak{L}_\lambda^\alpha$

$$\int \frac{d\Delta_1 \mathfrak{L}_\lambda^\alpha \, d\Delta_2 \mathfrak{L}_\lambda^\beta}{d\mathfrak{R}_0} = \begin{cases} \Delta_{12} r_0^{(\alpha)}(\lambda) & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}. \tag{25}$$

From the expansion of B in terms of the characteristic forms, we obtain the expression of A in terms of $\mathfrak{L}_\lambda^\alpha$

$$a^{(\alpha,\beta)}(E, E') = \int_\bullet \lambda \frac{d_\lambda \mathfrak{L}_E'^\alpha \, d_\lambda \mathfrak{L}_{E'}'^\beta}{d\mathfrak{R}_0}. \tag{26}$$

The results may be summarized in the following theorem.

THEOREM 3. *If the matrix* $A = (a^{(\alpha,\beta)}(E, E'))$ *is symmetric, limited with respect to* $\mathfrak{A}_0$, *and not identically zero, there exists a spectrum, lying within a finite interval of the* λ*-axis, which consists of characteristic numbers, or of a continuous spectrum, or of both, and for which the equations* (24) *have solutions, integrable* $H(\mathfrak{A}_0)$ *for every* α *and* λ. *The solutions form an orthogonal system* (25), *and the matrix* A *may be expressed in terms of them in the form* (26).

By applying results obtained by us* for linear equations with an unsymmetric matrix of coefficients, we obtain the following theorem.

THEOREM 4. *If the limited matrix* $A = (a^{(\alpha,\beta)}(E, E'))$ *is not identically zero, and there exists a limited symmetric matrix* $T = (t^{(\alpha,\beta)}(E, E'))$ *such that*

$$T(\mathfrak{F}, \mathfrak{F}) \geqq 0$$

the equality sign holding only for $\mathfrak{F} \equiv 0$, *and such that the product matrix*

$$C = AT$$

is symmetric, then there exists associated with A *a spectrum, lying within a finite interval of the* λ*-axis, which consists of characteristic numbers, or of a continuous spectrum, or of both, and for which the equation*

$$\int_\Delta \lambda d\mathfrak{L}_\lambda^\alpha = A[\Delta \mathfrak{L}_\lambda^\alpha]$$

has solutions.

Let $s_{ik} = T(\mathfrak{P}_i, \mathfrak{P}_k)$, then BS is a symmetric limited matrix, and the necessary conditions on B are fulfilled.

6. General orthogonal matrices

In this section we define a general orthogonal matrix $L = (l^{(\alpha,\beta)}(E, E'))$, and derive some properties. The orthogonal matrices represented by the relations (1), (4), (21), and (25) are special cases.

Let the functions $l^{(\alpha,\beta)}(E, E')$ be absolutely additive in E and E'; $a_0^{(\alpha)}(E)$ and $b_0^{(\beta)}(E')$ absolutely additive and monotonic; $\mathfrak{L}_\Sigma^\alpha$ integrable $H(\mathfrak{B}_0)$ for

* Pell, A. J., *Linear equations with unsymmetric systems of coefficients*, these Transactions, January, 1919.

every α and E. The matrix L forms an *orthogonal matrix with respect to* $\mathfrak{A}_0$, if

$$LL' = \Delta(\mathfrak{A}_0). \tag{27}$$

From

$$\int \frac{\left(d\left\{\mathfrak{F} - \sum_{\alpha=1}^{m}\sum_{i=1}^{n}\frac{\mathfrak{L}^{\alpha}_{E_i} L^{\alpha}_{E_i}[\mathfrak{F}]}{a_0^{(\alpha)}(E_i)}\right\}\right)^2}{d\mathfrak{B}_0} \geqq 0,$$

it follows that for every $\mathfrak{F}$ integrable $H(\mathfrak{B}_0)$

$$\sum_{\alpha=1}^{m}\sum_{i=1}^{n}\frac{(L^{\alpha}_{E_i}[\mathfrak{F}])^2}{a_0^{(\alpha)}(E_i)} \leqq \int \frac{(d\mathfrak{F})^2}{d\mathfrak{B}_0}.$$

From this, Bessel's inequality is obtained,

$$\int \frac{(dL[\mathfrak{F}])^2}{d\mathfrak{A}_0} \leqq \int \frac{(d\mathfrak{F})^2}{d\mathfrak{B}_0},$$

which expresses that $L[\mathfrak{F}]$ is integrable $H(\mathfrak{A}_0)$. As a special case we have that $\mathfrak{L}'^{\alpha}_{E}$ is integrable $H(\mathfrak{A}_0)$ for every α and E, therefore

$$L'[L[\mathfrak{F}]]$$

exists for every $\mathfrak{F}$ integrable $H(\mathfrak{B}_0)$.

The system $\mathfrak{L}^{\alpha}_{E}$ is *closed* if there exists no system $\mathfrak{G}$, not identically zero, integrable $H(\mathfrak{B}_0)$, and such that

$$L^{\alpha}_{E}[\mathfrak{G}] = 0$$

for every α and E.

If the matrix L is orthogonal and $\mathfrak{L}^{\alpha}_{E}$ is closed, then for every $\mathfrak{F}$ integrable $H(\mathfrak{B}_0)$

$$L'[L[\mathfrak{F}]] = \mathfrak{F},$$

and also

$$L'L = \Delta(\mathfrak{B}_0). \tag{28}$$

If the matrix L is composed of l_{ik}, $p_i^{(\alpha)}(s)$, $\pi_k^{(\beta)}(t)$, and $\Lambda^{(\alpha,\beta)}(s,t)$, the orthogonal condition (27) becomes

$$\sum_j l_{ij}\, l_{kj} + \sum_\beta \int_a^b \frac{d\pi_i^{(\beta)}\, d\pi_k^{(\beta)}}{db_0^{(\beta)}} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases},$$

$$\sum_j \Delta_1 p_j^{(\alpha_1)}(s)\, \Delta_2 p_j^{(\alpha_2)}(s) + \sum_\beta \int_a^b \frac{dt\, \Delta_1 \Lambda^{(\alpha_1,\beta)}(s,t)\, d_t\, \Delta_2 \Lambda^{(\alpha_2,\beta)}(s,t)}{db_0^{(\beta)}(t)}$$
$$= \begin{cases} \Delta_{12}\, a_0^{(\alpha_1)} & \alpha_1 = \alpha_2 \\ 0 & \alpha_1 \neq \alpha_2 \end{cases},$$

$$\sum_j l_{ij}\, \Delta p_j^{(\beta)}(s) + \sum_\beta \int_a^b \frac{d_t\, \Delta\Lambda^{(\alpha,\beta)}(s,t)\, d\pi_i^{(\beta)}(t)}{db_0^{(\beta)}(t)} = 0,$$

and (28) has a similar form.

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Page 203. O. E. GLENN. *On a new treatment of theorems of finiteness.*
Page 204, line 25, for "T" read "τ."
Page 205, equation 6, for "$\prod$" read "$\sum$."